arXiv:1302.2307v1 [math.CA] 10 Feb 2013

A Class of Extended Hypergeometric Functions and Its Applications

Luo Minjie

Department of Applied Mathematics, Donghua University, Shanghai 201620, PR China

E-mail: Mathwinnie@myopera.com

Abstract

Recently, there emerges different versions of beta function and hypergeometric functions containing extra parameters. Gaining enlightenment from these ideas, we will first introduce a new extension of generalized hypergeometric function and then put forward some fundamental results in the paper. Next, we will derive some properties of certain functions like extended Gauss hypergeometric functions, extended Appell's hypergeometric functions $\mathbf{F}_1^{(\kappa_l)}$, $\mathbf{F}_2^{(\kappa_l)}$, and extended Lauricella's hypergeometric function $\mathbf{F}_{D,(\kappa_l)}^{(r)}$,$\mathbf{F}_{A,(\kappa_l)}^{(r)}$, including transformation formulas, finite sum representations, Mellin-Barnes type integral representations and recurrence relations. Moreover, by using some new integral representation which will be presented in this paper, a Hardy-Hilbert type inequality involving extended Gauss hypergeometric functions will be established.

Keywords: Extended beta function, Generalized hypergeometric functions, Appell's hypergeometric functions, Lauricella's hypergeometric functions, Ramanujan's Master Theorem, Method of brackets.

1. Introduction and preliminaries.

In the eighteenth century, L. Euler (1707-1783) concerned himself with the problem of interpolating between the numbers

$$n! = \int_0^\infty e^{-t} t^n dt, \quad n = 0, 1, 2, \cdots$$

with nonintergral values of n. This problem led Euler in 1729 to the now famous gamma function, a generalization of the factorial function that gives meaning to $x!$ when x is any positive number. His result can be extended to certain negative numbers and even to complex numbers [2]. The integral representation of now widely accepted gamma function $\Gamma(x)$ is

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \ \Re(x) > 0. \tag{1}$$

In 1994, by inserting a regularization factor $e^{-bt^{-1}}$, Chaudhry and Zubair [5] have introduced the following extension of gamma function

$$\Gamma_p(x) = \int_0^\infty t^{x-1} \exp\left(-t - \frac{b}{t}\right) dt, \ \Re(b) > 0. \tag{2}$$

For $\Re(b) > 0$ this factor removes the singularity coming from the $t = 0$ limit and for $b = 0$ reduces to the original gamma function defined by (1). In 1997, Chaudhry et al. [6] presented the following extension of Euler's beta function

$$B_b(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \exp\left(-\frac{b}{t(1-t)}\right) dt, \ \Re(b) > 0. \tag{3}$$

Afterwards, Chaudhry et al. [5] used $B_b(x, y)$ to extended Gauss hypergeometric function and Kummer confluent hypergeometric function as follows

$$F_b(a, b; c; z) = \sum_{m=0}^\infty (a)_m \frac{B_b(b+m, c-b)}{B(b, c-b)} \frac{z^m}{m!}, \quad (b \geq 0; \ |z| < 1; \ \Re(c) > \Re(b) > 0) \tag{4}$$

$$\Phi_b(b;c;z)=\sum_{m=0}^{\infty}\frac{B_b(b+m,c-b)}{B(b,c-b)}\frac{z^m}{m!},\quad (b\geq 0;\ \Re(c)>\Re(b)>0) \tag{5}$$

where $(a)_m$ denotes the Pochhammer symbol defined, in terms of gamma functions, by

$$(a)_m=\frac{\Gamma(a+m)}{\Gamma(a)}=\begin{cases}1 & m=0;\ a\in\mathbb{C}\backslash\{0\}\\ a(a+1)(a+2)\cdots(a+m-1) & m\in\mathbb{N};\ a\in\mathbb{C}.\end{cases}$$

For $b=0$, functions (4) and (5) reduces to the usual hypergeometric functions.

In a similar manner, M. Ali Ozarslan and E. Ozergin [21] defined the extension of Appell's functions $F_1(a,b,c;d;x,y;b)$, $F_2(a,b,c;d,e;x,y;b)$ and extended Lauricella's hypergeometric function $\mathbf{F}_{D,b}^{(3)}(a,b,c,d;e;x,y,z)$.

In 2011, E. Ozergin, M. Ali Ozarslan, and A. Altin [22] introducing the following generalizations

$$\Gamma_b^{(\alpha,\beta)}(x)=\int_0^{\infty}t^{x-1}{}_1F_1\left(\alpha;\beta;-t-\frac{b}{t}\right)dt\ \ (\Re(\alpha)>0,\ \Re(\beta)>0,\Re(p)>0,\Re(x)>0) \tag{6}$$

$$B_b^{(\alpha,\beta)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-b}{t(1-t)}\right)dt. \tag{7}$$
$$(\Re(\alpha)>0,\ \Re(\beta)>0,\Re(p)>0,\Re(x)>0,\Re(y)>0)$$

In this paper, our extension mainly based on using the following generalization of gamma and beta functions.

Definition 1.1. *[18, p.243]: Let a function* $\Theta\left(\{\kappa_l\}_{l\in\mathbb{N}_0};z\right)$ *be analytic within the disk* $|z|<R$ $(0<R<\infty)$ *and let its Taylor-Maclaurin coeficients be explicitly deonted by the sequence* $\{\kappa_l\}_{l\in\mathbb{N}_0}$. *Suppose also that the function* $\Theta\left(\{\kappa_l\}_{l\in\mathbb{N}_0};z\right)$ *can be continued analytically in the right half-plane* $\Re(z)>0$ *with the asymptotic property given as follows:*

$$\Theta(\kappa_l;z)\equiv\Theta\left(\{\kappa_l\}_{l\in\mathbb{N}_0};z\right)=\begin{cases}\sum_{l=0}^{\infty}\kappa_l\frac{z^l}{l!} & (|z|<R;\ 0<R<\infty;\kappa_0=1)\\ M_0 z^{\omega}\exp(z)\left[1+O\left(\frac{1}{z}\right)\right] & (\Re(z)\to\infty;M_0>0;\omega\in\mathbb{C})\end{cases} \tag{8}$$

for some suitable constants M_0 *and* ω *depending essentially on the sequence* $\{\kappa_l\}_{l\in\mathbb{N}_0}$. *We can define extended Gamma function* $\Gamma_b^{(\kappa_l)}(z)$ *and the extended Beta function* $\mathcal{B}^{(\kappa_l)}(\alpha,\beta;b)$ *by*

$$\Gamma_b^{(\kappa_l)}(z)=\int_0^{\infty}t^{z-1}\Theta\left(\{\kappa_l\};-t-\frac{b}{t}\right)dt\ \ (\Re(z)>0;\ \Re(b)\geq 0) \tag{9}$$

and

$$\mathcal{B}_b^{(\kappa_l)}(\alpha,\beta;b)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\Theta\left(\{\kappa_l\};-\frac{b}{t(1-t)}\right)dt \tag{10}$$
$$(\min\{\Re(\alpha),\Re(\beta)\}>0;\ \Re(b)\geq 0).$$

By introducing one additional parameter d *with* $\Re(d)\geq 0$, *we have*

$$\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\Theta\left(\{\kappa_l\};-\frac{b}{t}-\frac{d}{1-t}\right)dt \tag{11}$$
$$(\min\{\Re(\alpha),\Re(\beta)\}>0;\ \min\{\Re(b),\Re(d)\}\geq 0)$$

Remark

(I) For given sequence $\{\kappa_l\}_{l\in\mathbb{N}_0}$ our definitions would reduce to known or new extensions of the Gamma and Beta functions. Recall that the asymptotic behavior of Kummer's confluent hypergeometric function at infinity has the form

$$\Phi(a;c;z)={}_1F_1(a;c;z)=\frac{\Gamma(c)}{\Gamma(a)}e^z z^{a-c}\left[1+O\left(\frac{1}{z}\right)\right],\ \Re(z)\to\infty.$$

It is clear that $\Phi(a;c;z)$ is a special case of $\Theta(\kappa_l;z)$ with sequence $\left\{\frac{(a)_l}{(c)_l}\right\}_{l\in\mathbb{N}_0}$. And our Gamma function $\Gamma_b^{(\kappa_l)}$ and Beta function $\mathcal{B}^{(\kappa_l)}$ are reduced to (6) and (7) which has been studied in [22]. We

can also replace ${}_1F_1(a;c;z)$ with $\exp(z)$ to get more basic extension of Gamma and Beta function. (see, for details, [7])

(II) Function $\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha,\beta)$ is a further extension of Beta function $\mathcal{B}_{b}^{(\kappa_l)}(\alpha,\beta)$ by introducing complex parameter d ($\Re(d)\geq 0$). This improvement may be essential since $-\frac{b}{t}-\frac{d}{1-t}=-\left(\frac{b+(d-b)t}{t(1-t)}\right)$. Note that if we set $d=b$ we will get function $\mathcal{B}_{b}^{(\kappa_l)}(\alpha,\beta)$. In general, form $-\frac{b}{t}-\frac{d}{1-t}$ may give us more flexibility for making transformations.

$\diamondsuit$

By using $\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha,\beta)$ we can get the corresponding further extension of Gauss hypergeometric function as follows.

Definition 1.2. *[18] The Extended Gauss hypergeometric function ${}_2F_1^{(\kappa_l)}$ is defined by*

$$ {}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1,\alpha_2\\ \beta_1\end{array};z;b,d\right]=\sum_{n=0}^{\infty}(\alpha_1)_n\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_2+n,\beta_1-\alpha_2)}{B(\alpha_2,\beta_1-\alpha_2)}\frac{z^n}{n!}. \tag{12}$$

$$(|z|<1;\Re(c)>\Re(b)>0;\min\{\Re(b),\Re(d)\}\geq 0)$$

If $\Theta(\kappa_l;z)=\exp(z)$, we write

$$ {}_2F_1\left[\begin{array}{c}\alpha_1,\alpha_2\\ \beta_1\end{array};z;b,d\right]=\sum_{n=0}^{\infty}(\alpha_1)_n\frac{\mathcal{B}_{b,d}(\alpha_2+n,\beta_1-\alpha_2)}{B(\alpha_2,\beta_1-\alpha_2)}\frac{z^n}{n!}.$$

Its integral representation can be easily derived.

Theorem 1.1. *For the extended Gauss hypergeometric function ${}_2F_1^{(\kappa_l)}(\alpha_1,\alpha_2;\beta_1;z;b,d)$, we have the following integral represetation:*

$$ {}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1,\alpha_2\\ \beta_1\end{array};z;b,d\right]=\frac{1}{B(\alpha_2,\beta_1-\alpha_2)}\int_0^1 t^{\alpha_2-1}(1-t)^{\beta_1-\alpha_2-1}(1-zt)^{-\alpha_1}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)dt, \tag{13}$$

$$\Re(\beta_1)>\Re(\alpha_2)>0;\ \Re(b),\Re(d)>0;b=d=0,|\arg(1-z)<\pi|$$

To prove this integral representation, we first expand $(1-zt)^{-\alpha_1}$ as its Taylor series

$$(1-zt)^{-\alpha_1}=\sum_{n=0}^{\infty}(\alpha_1)_n\frac{(zt)^n}{n!}.$$

It is clear that this series is absolutely and uniformly convergent. Thus, we can interchange the order of summation and than integrate out t to get the result.

As outlined above, we have introduced the general idea of the development path of the subject, the extended Beta function and its application in generalizing some hypergeometric series. As we all know, general hypergeometric functions have important applications in mathematical, physical and chemical areas. Thus, quite naturally, we should not neglect the question that whether we could extend the definition of generalized hypergeometric function by using extended beta function $\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha,\beta)$. Or whether we could prove that general hypergeometric functions are specific cases of the functions we get in the paper. The significance of such questions lies in the fact that a large number of practical problems require more complex hypergeometric functions of complicated coefficients. Fortunately, the answers are almost certainly "yes", inspired by the way how extended Appell's hypergeometric functions $\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y;b,d)$ are constructed in [18], [21]. In fact, the method can also be applied to ratiocinate a new kind of multivariate hypergeometric function, i.e., $\mathbf{F}_{A,(\kappa_l)}^{(r)}(\alpha,\beta_1,\cdots,\beta_r;\gamma_1,\cdots\gamma_r;x_1,\cdots,x_r;b,d)$, with its large numbers of formulas, which will be analyzed in section 3.

Let us review some basic conclusions concerning with general hypergeometric functions. The generalized

hypergeometric function with p numerator and q denominator parameters is defined by [13, p. 27]

$$ {}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z) = {}_pF_q\left[\begin{array}{ccc}\alpha_1 & \cdots & \alpha_p \\ \beta_1 & \cdots & \beta_q\end{array};z\right] = \sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{z^k}{k!}, \tag{14} $$

$$ = \frac{\Gamma(\beta_1)\cdots\Gamma(\beta_q)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+k)\cdots\Gamma(\alpha_p+k)}{\Gamma(\beta_1+k)\cdots\Gamma(\beta_q+k)}\frac{z^k}{k!} \tag{15} $$

$$ (\alpha_l,\beta_j\in\mathbb{C},\beta_j\neq 0,-1,-2,\cdots,l=1,\cdots,p;j=1,\cdots,q) $$

which is absolutely convergent for all values of $z\in\mathbb{C}$ if $p\leq q$. When $p=q+1$, the series is absolutely convergent for $|z|<1$ and for $|z|=1$ when $\Re\left(\sum_{j=1}^{q}\beta_j-\sum_{l=1}^{p}\alpha_l\right)>0$, while it is conditionally convergent for $|z|=1$ $(z\neq 1)$ if $-1<\Re\left(\sum_{j=1}^{q}\beta_j-\sum_{l=1}^{p}\alpha_l\right)\leq 0$. More detailed information may be found in [2, 3]and [24].

In what follows, we consider the extended generalized hypergeometric function ${}_pF_q^{(\kappa_l)}(\alpha_1,\cdots\alpha_p;\beta_1\cdots\beta_q;z;b,d)$ defined by

$$ {}_pF_q^{(\kappa_l)}\left[\begin{array}{ccc}\alpha_1 & \cdots & \alpha_p \\ \beta_1 & \cdots & \beta_q\end{array};z;b,d\right] = \begin{cases} \displaystyle\sum_{m=0}^{\infty}(\alpha_1)_m\prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_{j+1}+m,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1},\beta_j-\alpha_{j+1})}\frac{z^m}{m!}, & |z|<1;\ p=q+1 \\ \displaystyle\sum_{m=0}^{\infty}\prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_j+m,\beta_j-\alpha_j)}{B(\alpha_j,\beta_j-\alpha_j)}\frac{z^m}{m!}, & p=q \\ \displaystyle\sum_{m=0}^{\infty}\frac{1}{(\beta_1)_m}\cdots\frac{1}{(\beta_r)_m}\prod_{j=1}^{p}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_j+m,\beta_{r+j}-\alpha_j)}{B(\alpha_j,\beta_{r+j}-\alpha_j)}\frac{z^m}{m!}, & r=q-p,\ p<q. \end{cases} \tag{16} $$

If we set $b=d=0$, then

$$ {}_pF_q^{(\kappa_l)}(\alpha_1,\cdots\alpha_p;\beta_1\cdots\beta_q;z;b,d) = {}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z). $$

If $p=2,\ q=1$ then ${}_pF_q^{(\kappa_l)}$ reduces to extended Gauss hypergeometric function (12).

The paper is organized as follows.

In section 2, we will prove Euler type integral representation and expound some properties of new extended generalized hypergeometric function ${}_pF_q^{(\kappa_l)}$. By using *Ramanujan's Master Theorem*, we can get the Mellin-Barnes type integral representation for extended generalized hypergeometric function ${}_pF_q^{(\kappa_l)}$. Then, as a special case of ${}_pF_q^{(\kappa_l)}$, more properties of extended Gauss hypergeometric functions such as differentiation formulas and recurrence formulas will be discussed in detail. In the last part of this section, we give an improved definition, **definition 2.2**. By doing this, we are not only able to prove a summation theorem deserving special attention, but also to establish the relations between modified functions and fractional operators.

In section 3, we will give more integral identities and reduction formulas for functions such as extended Appell's hypergeometric functions $\mathbf{F}_1^{(\kappa_l)}(\alpha,\beta,\beta';\gamma;x,y;b,d)$, $\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta,\beta';\gamma,\gamma';x,y;b,d)$ and Laucrella's hypergeometric functions $\mathbf{F}_{D,(\kappa_l)}^{(r)}$, defined in [18]. In addition, we give a definition of a new type of multivariate hypergeometric function, i.e.,

$$ \mathbf{F}_{A,(\kappa_l)}^{(r)}(\alpha,\beta_1,\cdots,\beta_r;\gamma_1,\cdots\gamma_r;x_1,\cdots,x_r;b,d) = \sum_{m_1,\cdots,m_r=0}^{\infty}(\alpha)_{m_1+\cdots+m_r}\prod_{j=1}^{r}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\beta_j+m_j,\gamma_j-\beta_j)}{B(\beta_j,\gamma_j-\beta_j)}\frac{x_1^{m_1}\cdots x_r^{m_r}}{m_1!\cdots m_r!} \tag{17} $$

$$ (|x_1|+\cdots+|x_r|<1;\ \min\{\Re(b),\Re(d)\}\geq 0) $$

and draw conclusions about it on the basis of the definition. Viewed as a a generalization of *Ramanujan's Master Theorem* and with its wide application in evaluating multidimensional definite integrals and the

Feynman integral, the *Method of Bracket* is also used to prove the Mellin-Barnes integral representations of these multivariate hypergeometric functions.

In section 4, we reestablish a Hilbert-Hardy inequality by using extended Gauss hypergeometric functions. It is remarkable that the extended Gauss hypergeometric functions can also be used to generalize other inequalities.

2. Extended Generalized Hypergeometric Functions

We first give a formal definition and some remarks for extended generalized hypergeometric functions.

Definition 2.1. *For suitably constrained (real or complex) parameters* $\alpha_j,\ j=1,\cdots p$*;* $\beta_i,\ i=1,\cdots,q$*, we define the extended generalized hypergeometric functions by*

$$
{}_pF_q^{(\kappa_l)}\left[\begin{matrix}\alpha_1 & \cdots & \alpha_p\\ \beta_1 & \cdots & \beta_q\end{matrix};z;b,d\right]=
\begin{cases}
\displaystyle\sum_{m=0}^{\infty}(\alpha_1)_m\prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_{j+1}+m,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1},\beta_j-\alpha_{j+1})}\frac{z^m}{m!}, & |z|<1;\ p=q+1;\Re(\beta_j)>\Re(\alpha_{j+1})>0\\
\displaystyle\sum_{m=0}^{\infty}\prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_j+m,\beta_j-\alpha_j)}{B(\alpha_j,\beta_j-\alpha_j)}\frac{z^m}{m!}, & z\in\mathbb{C};p=q;\Re(\beta_j)>\Re(\alpha_j)>0\\
\displaystyle\sum_{m=0}^{\infty}\frac{1}{(\beta_1)_m}\cdots\frac{1}{(\beta_r)_m}\prod_{j=1}^{p}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_j+m,\beta_{r+j}-\alpha_j)}{B(\alpha_j,\beta_{r+j}-\alpha_j)}\frac{z^m}{m!}, & z\in\mathbb{C};r=q-p,\ p<q;\Re(\beta_{r+j}>\Re(\alpha_j)>0)
\end{cases}
\tag{18}
$$

Remark

(I) If the numerator parameter α_1 in the first series of (18) is zero or a negative integer, the series terminates, i.e.,

$$
{}_{q+1}F_q^{(\kappa_l)}\left[\begin{matrix}-n & \cdots & \alpha_{q+1}\\ \beta_1 & \cdots & \beta_q\end{matrix};z;b,d\right]=\sum_{m=0}^{n}(-n)_m\prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_{j+1}+m,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1},\beta_j-\alpha_{j+1})}\frac{z^m}{m!}.
$$

And we should make sure that no denominator parameter $\beta_1,\cdots,\beta_r$ is allowed to be zero or a negative integer in the third expression of (18).

(II) The convergence of series ${}_pF_q^{(\kappa_l)}$ with different parameter p and q can be obtained by using Comparison Test and the convergence of known corresponding generalized hypergeometric functions ${}_pF_q$. Their own convergence region is therefore clear.

(III) Note that for usual generalized hypergeometric function ${}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z)$, its numerator-parameters $\alpha_1,\cdots\alpha_p$ and denominator-parameters $\beta_1,\cdots,\beta_q$ can be arbitrarily rearranged without any effects on its value. But under the setting of our definition, we need to point out that the expression shown on the right-hand side of (18) is closely associated with the order of the parameters given in the square bracket on the the left-hand side of (18).

(IV) Although the general definition seems complicated, it will be very convenient in practice.

$\diamondsuit$

We will first discuss in detail some fundamental properties of function (18) and then in the final part of this section a modified definition will be presented.

2.1. ***Some basic results about the extended generalized hypergeometric function***

What we most concerned about is the Euler type integral representation of our new function (18). The following theorem aims to demonstrate that the form of the Euler type integral representation of ${}_pF_q^{(\kappa_l)}$ is very similar to that of the Euler type integral representation of ${}_pF_q$.

Theorem 2.1. *If* $p \leq q+1$, $\Re(\beta_q) > \Re(\alpha_p) > 0$ *and* $\Re(b) > 0$, $\Re(d) > 0$, *we have*

$$\begin{aligned}{}_pF_q^{(\kappa_l)}\left[\begin{array}{ccc}\alpha_1 & \cdots & \alpha_p\\ \beta_1 & \cdots & \beta_q\end{array};z;b,d\right] &= \frac{\Gamma(\beta_q)}{\Gamma(\alpha_p)\Gamma(\beta_q-\alpha_p)}\int_0^1 t^{\alpha_p-1}(1-t)^{\beta_q-\alpha_p-1}\\ &\times {}_{p-1}F_{q-1}^{(\kappa_l)}\left[\begin{array}{ccc}\alpha_1 & \cdots & \alpha_{p-1}\\ \beta_1 & \cdots & \beta_{q-1}\end{array};zt;b,d\right]\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)dt. \qquad (19)\end{aligned}$$

Eq (19) also holds when $b = d = 0$, *proided that* $|\arg(1-z) < \pi|$.

Proof

We need to verify that formula (19) holds for three different expressions of ${}_pF_q^{(\kappa_l)}(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z;b,d)$ given in (18), respectively. Consider the case $p = q+1$. Note that

$$\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_{q+1}+m,\beta_q-\alpha_{q+1})}{B(\alpha_{q+1},\beta_q-\alpha_{q+1})} = \frac{\Gamma(\beta_q)}{\Gamma(\alpha_{q+1})\Gamma(\beta_q-\alpha_{q+1})}\int_0^1 t^{\alpha_{q+1}+m-1}(1-t)^{\beta_q-\alpha_{q+1}-1}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)dt.$$

Substituting this in expression (18), we get

$$\begin{aligned}{}_{q+1}F_q^{(\kappa_l)}\left[\begin{array}{ccc}\alpha_1 & \cdots & \alpha_{q+1}\\ \beta_1 & \cdots & \beta_q\end{array};z;b,d\right] &= \sum_{m=0}^{\infty}(\alpha_1)_m\prod_{j=1}^{q-1}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_{j+1}+m,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1},\beta_j-\alpha_{j+1})}\frac{\Gamma(\beta_q)}{\Gamma(\alpha_{q+1})\Gamma(\beta_q-\alpha_{q+1})}\\ &\int_0^1 t^{\alpha_{q+1}+m-1}(1-t)^{\beta_q-\alpha_{q+1}-1}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)dt\cdot\frac{z^m}{m!}\\ &= \frac{\Gamma(\beta_q)}{\Gamma(\alpha_{q+1})\Gamma(\beta_q-\alpha_{q+1})}\int_0^1 t^{\alpha_{q+1}-1}(1-t)^{\beta_q-\alpha_{q+1}-1}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)\\ &\times\sum_{m=0}^{\infty}(\alpha_1)_m\prod_{j=1}^{q-1}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_{j+1}+m,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1},\beta_j-\alpha_{j+1})}\frac{(zt)^m}{m!}dt\\ &= \frac{\Gamma(\beta_q)}{\Gamma(\alpha_{q+1})\Gamma(\beta_q-\alpha_{q+1})}\int_0^1 t^{\alpha_{q+1}-1}(1-t)^{\beta_q-\alpha_{q+1}-1}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)\\ &\times {}_qF_{q-1}^{(\kappa_l)}\left[\begin{array}{ccc}\alpha_1 & \cdots & \alpha_q\\ \beta_1 & \cdots & \beta_{q-1}\end{array};zt;b,d\right]dt. \qquad (20)\end{aligned}$$

For $p = q$, we have

$$\begin{aligned}{}_qF_q^{(\kappa_l)}\left[\begin{array}{ccc}\alpha_1 & \cdots & \alpha_q\\ \beta_1 & \cdots & \beta_q\end{array};z;b,d\right] &= \frac{\Gamma(\beta_q)}{\Gamma(\alpha_q)\Gamma(\beta_q-\alpha_q)}\int_0^1 t^{\alpha_q-1}(1-t)^{\beta_q-\alpha_q-1}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)\\ &\times {}_{q-1}F_{q-1}^{(\kappa_l)}\left[\begin{array}{ccc}\alpha_1 & \cdots & \alpha_{q-1}\\ \beta_1 & \cdots & \beta_{q-1}\end{array};zt;b,d\right]dt.\end{aligned}$$

It is clear that this relation is valid for $p < q$. This completes the proof. $\square$

Remark

A multidimensional case of Euler type integral representation of (20) is given by:

$$\begin{aligned}{}_{q+1}F_q^{(\kappa_l)}\left[\begin{array}{ccc}\alpha_1 & \cdots & \alpha_{q+1}\\ \beta_1 & \cdots & \beta_q\end{array};z;b,d\right] &= \prod_{j=1}^{q}\frac{\Gamma(\beta_j)}{\Gamma(\alpha_{j+1})\Gamma(\beta_j-\alpha_{j+1})}\\ \times\int_0^1\cdots\int_0^1\prod_{j=1}^{q}&\left\{t_j^{\alpha_{j+1}}(1-t_j)^{\beta_j-\alpha_{j+1}-1}\Theta\left(\kappa_l;-\frac{b}{t_j}-\frac{d}{1-t_j}\right)\right\}(1-t_1t_2\cdots t_qz)^{-\alpha_1}dt_1\cdots dt_q,\end{aligned}$$

which follows from the repeated application of the functional equation (20). If we set $b = d = 0$, then above equation reduces to the form given by [12, p.132, Eq.(4.2)] $\diamondsuit$

Theorem 2.2. *For* $p \leq q+1$, *we have the following differentiation formula:*

$$\frac{d^n}{dz^n}\left\{{}_pF_q^{(\kappa_l)}\left[\begin{array}{ccc}\alpha_1 & \cdots & \alpha_p\\ \beta_1 & \cdots & \beta_q\end{array};z;b,d\right]\right\} = \frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}{}_pF_q^{(\kappa_l)}\left[\begin{array}{ccc}\alpha_1+n & \cdots & \alpha_p+n\\ \beta_1+n & \cdots & \beta_q+n\end{array};z;b,d\right] \qquad (21)$$

Proof

Taking the derivative of ${}_{q+1}F_q^{(\kappa_l)}$ with respect to z, we obtain

$$\frac{d}{dz}\left\{{}_{q+1}F_q^{(\kappa_l)}\left[\begin{array}{ccc}\alpha_1 & \cdots & \alpha_p\\ \beta_1 & \cdots & \beta_q\end{array};z;b,d\right]\right\}=\frac{d}{dz}\left\{\sum_{m=0}^{\infty}(\alpha_1)_m\prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_{j+1}+m,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1},\beta_j-\alpha_{j+1})}\frac{z^m}{m!}\right\}$$
$$=\sum_{m=1}^{\infty}(\alpha_1)_m\prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_{j+1}+m,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1},\beta_j-\alpha_{j+1})}\frac{z^{m-1}}{(m-1)!}.$$

Replacing $m\to m+1$ we get

$$\frac{d}{dz}\left\{{}_{q+1}F_q^{(\kappa_l)}\left[\begin{array}{ccc}\alpha_1 & \cdots & \alpha_{q+1}\\ \beta_1 & \cdots & \beta_q\end{array};z;b,d\right]\right\}=\sum_{m=0}^{\infty}(\alpha_1)_{m+1}\prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_{j+1}+m+1,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1},\beta_j-\alpha_{j+1})}\frac{z^m}{m!}$$
$$=\alpha_1\frac{\prod_{j=1}^{q}\alpha_{j+1}}{\prod_{j=1}^{q}\beta_j}\sum_{m=0}^{\infty}(\alpha_1+1)\prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_{j+1}+1+m,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1}+1,\beta_j-\alpha_{j+1})}\frac{z^m}{m!}$$
$$=\alpha_1\frac{\prod_{j=1}^{q}\alpha_{j+1}}{\prod_{j=1}^{q}\beta_j}\,{}_{q+1}F_q^{(\kappa_l)}\left[\begin{array}{ccc}\alpha_1+1 & \cdots & \alpha_{q+1}+1\\ \beta_1+1 & \cdots & \beta_q+1\end{array};z;b,d\right].$$

Recursive application of this procedure gives us the general form

$$\frac{d^n}{dz^n}\left\{{}_{q+1}F_q^{(\kappa_l)}\left[\begin{array}{ccc}\alpha_1 & \cdots & \alpha_{q+1}\\ \beta_1 & \cdots & \beta_q\end{array};z;b,d\right]\right\}=\frac{(\alpha_1)_n\cdots(\alpha_{q+1})_n}{(\beta_1)_n\cdots(\beta_q)_n}\,{}_{q+1}F_q^{(\kappa_l)}\left[\begin{array}{ccc}\alpha_1+n & \cdots & \alpha_{q+1}+n\\ \beta_1+n & \cdots & \beta_q+n\end{array};z;b,d\right].$$

Similarly, we can prove this result for the case $p\le q$. □

For $p=2$ and $q=1$ we get the following corollary.

Corollary 1.

$$\frac{d^n}{dz^n}\left\{{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1,\alpha_2\\ \beta_1\end{array};z;b,d\right]\right\}=\frac{(\alpha_1)_n(\alpha_2)_n}{(\beta_1)_n}\,{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1+n,\alpha_2+n\\ \beta_1+n\end{array};z;b,d\right]\tag{22}$$

Remark

If we take $\Theta(\kappa_l;z)={}_1F_1(\alpha;\beta;z)$, $b=d$, then equation (21) reduces to

$$\frac{d^n}{dz^n}\left\{F_b^{(\alpha,\beta)}\left[\begin{array}{c}a,b\\ c\end{array};z\right]\right\}=\frac{(b)_n(a)_n}{(c)_n}F_b^{(\alpha,\beta)}\left[\begin{array}{c}a+n,b+n\\ c+n\end{array};z\right]$$

given in [22, Theorem 3.3]. ♢

In order to derive the Mellin-Barnes type contour integral representation of ${}_pF_q^{(\kappa_l)}(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z;b,d)$ we need to introduce the following well-known theorem. It is widely used to evaluate definite integrals and infinite series.

Theorem 2.3 (Ramanujan's Master Theorem [1]). Assume f admits an expansion of the form:

$$f(x)=\sum_{k=0}^{\infty}\frac{\lambda(k)}{k!}(-x)^k.$$

Then, the Mellin transform of f is given by

$$\int_0^{\infty}x^{s-1}f(x)\,dx=\Gamma(s)\,\lambda(-s).$$

By means of Ramanujan's master theorem we obtain:

Theorem 2.4. *We have the following Mellin-Barnes type integral representation of function (18), namely,*

$$
{}_pF_q^{(\kappa_l)}\left[\begin{matrix}\alpha_1 & \cdots & \alpha_p\\ \beta_1 & \cdots & \beta_q\end{matrix};z;b,d\right]
=\begin{cases}
\dfrac{1}{2\pi i}\displaystyle\int_L \prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}\left(\alpha_{j+1}-s,\beta_j-\alpha_{j+1}\right)}{B\left(\alpha_{j+1},\beta_j-\alpha_{j+1}\right)}\frac{\Gamma(s)\Gamma(\alpha_1-s)}{\Gamma(\alpha_1)}(-z)^{-s}\,ds, & p=q+1\\[2ex]
\dfrac{1}{2\pi i}\displaystyle\int_L \prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}\left(\alpha_j-s,\beta_j-\alpha_j\right)}{B\left(\alpha_j,\beta_j-\alpha_j\right)}\Gamma(s)(-z)^{-s}\,ds, & p=q\\[2ex]
\dfrac{1}{2\pi i}\displaystyle\int_L \prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}\left(\alpha_j-s,\beta_{j+r}-\alpha_j\right)}{B\left(\alpha_j,\beta_{j+r}-\alpha_j\right)}\frac{\Gamma(\beta_1)\cdots\Gamma(\beta_r)\Gamma(s)}{\Gamma(\beta_1-s)\cdots\Gamma(\beta_r-s)}(-z)^{-s}\,ds, & r=q-p,\ p<q.
\end{cases}
\tag{23}
$$

where L *is a Barnes path of integration, that is,* L *starts at* $-i\infty$ *and runs to* $+i\infty$ *in the* s*-plane, curving if necessary to put the poles of* $\Gamma(\alpha_1-s)$ *to the left of the path and to put the poles of* $\Gamma(s)$ *to the right of the path.*

Proof

The result follows directly by using Ramanujan's Master Theorem and inverse Mellin transform. □

In section 3, a Multidimensional Ramanjuan's Master Theorem will be introduced to obtain double and multiple contour integral representations for extended Appell's and Lauricella's hypergeometric functions.

2.2. ***More properties about Gauss hypergeometric function***

Although many different kinds of extensions of Gauss hypergeometric function have been given, we still know little about their sorts of properties. Therefore, we list some formulas and conclusions about extended Gauss hypergeometric function,of which some are originated from classical theories and others are drawn on the basis of recent researches.

Theorem 2.5. *The following transformations hold true for extended Gauss hypergeometric function:*

1. Pfaff transformation

$$
{}_2F_1^{(\kappa_l)}\left[\begin{matrix}\alpha_1,\alpha_2\\ \beta_1\end{matrix};z;b,d\right]=(1-z)^{-\alpha_1}\,{}_2F_1^{(\kappa_l)}\left[\begin{matrix}\alpha_1,\beta_1-\alpha_2\\ \alpha_2\end{matrix};\frac{z}{1-z};d,b\right],\ \left(|\arg(1-z)|<\pi\right). \tag{24}
$$

2. Euler transformation *When* $\Theta(\kappa_l;z)=\exp(z)$*, we have:*

$$
{}_2F_1\left[\begin{matrix}\alpha_1,\alpha_2\\ \beta_1\end{matrix};z;b,d\right]=e^{-(1-z)b-zd}(1-z)^{\beta_1-\alpha_2-\alpha_1}\,{}_2F_1\left[\begin{matrix}\beta_1-\alpha_1,\beta_1-\alpha_2\\ \beta_1\end{matrix};z;\frac{b}{1-z},(1-z)d\right]. \tag{25}
$$

$$
\left(|\arg(1-z)|<\pi,\ \Re(z)<1\right)
$$

Proof

1. The proof of transformation (24) is direct. Since we have $[1-z(1-t)]^{-a}=(1-z)^{-a}\left(1+\frac{z}{1-z}t\right)^{-a}$ then, by replacing $t\to 1-t$, we obtain

$$
\begin{aligned}
{}_2F_1^{(\kappa_l)}\left[\begin{matrix}\alpha_1,\alpha_2\\ \beta_1\end{matrix};z;b,d\right]&=\frac{1}{B(\alpha_2,\beta_1-\alpha_2)}\int_0^1(1-t)^{\alpha_2-1}t^{\beta_1-\alpha_2-1}[1-z(1-t)]^{-\alpha_1}\,\Theta\left(\kappa_l;-\frac{b}{1-t}-\frac{d}{t}\right)dt\\
&=\frac{(1-z)^{-a}}{B(\alpha_2,\beta_1-\alpha_2)}\int_0^1 t^{\beta_1-\alpha_2-1}(1-t)^{\alpha_2-1}\left(1+\frac{z}{1-z}t\right)^{-\alpha_1}\Theta\left(\kappa_l;-\frac{d}{t}-\frac{b}{1-t}\right)dt\\
&=(1-z)^{-a}\,{}_2F_1^{(\kappa_l)}\left[\begin{matrix}\alpha_1,\beta_1-\alpha_2\\ \alpha_2\end{matrix};\frac{z}{1-z};d,b\right].
\end{aligned}
$$

2. When $\Theta(\kappa_l;z)=\exp(z)$, from Pfaff transformation (24) we have

$$
{}_2F_1\left[\begin{matrix}\alpha_1,\alpha_2\\ \beta_1\end{matrix};z;b,d\right]=(1-z)^{-\alpha_1}\,{}_2F_1\left[\begin{matrix}\alpha_1,\beta_1-\alpha_2\\ \alpha_2\end{matrix};\frac{z}{1-z};d,b\right].
$$

The rest of the proof will be presented in the **section 4** since we need more extra results built in that section. □

We also give a somewhat complicated differentiation formula for extended Gauss hypergeometric function.

Theorem 2.6. *One has*

$$\frac{d^n}{dz^n}\left\{z^{\alpha_1+n-1}{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1,\alpha_2\\ \beta_1\end{array};z;b,d\right]\right\}=(\alpha_1)_n\,z^{\alpha_1-1}{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1,\alpha_2\\ \beta_1\end{array};z;b,d\right]. \tag{26}$$

Proof

This formula can be obtained by direct computation,

$$\begin{aligned}\frac{d}{dz}\left\{z^{\alpha_1}{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1,\alpha_2\\ \beta_1\end{array};z;b,d\right]\right\}&=\frac{d}{dz}\left\{\sum_{n=0}^{\infty}(\alpha_1)_n\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_2+n,\beta_1-\alpha_2)}{B(\alpha_2,\beta_1-\alpha_2)}\frac{z^{n+\alpha_1}}{n!}\right\}\\&=\sum_{n=0}^{\infty}(\alpha_1)_n(\alpha_1+n)\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_2+n,\beta_1-\alpha_2)}{B(\alpha_2,\beta_1-\alpha_2)}\frac{z^{n+\alpha_1-1}}{n!}\\ (\text{Note that } (a+n)(a)_n=a(a+1)_n)&=\alpha_1 z^{\alpha_1-1}\sum_{n=0}^{\infty}(\alpha_1+1)_n\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_2+n,\beta_1-\alpha_2)}{B(\alpha_2,\beta_1-\alpha_2)}\frac{z^n}{n!}\\&=\alpha_1 z^{\alpha_1-1}{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1+1,\alpha_2\\ \beta_1\end{array};z;b,d\right].\end{aligned}$$

$$\begin{aligned}\frac{d^2}{dz^2}\left\{z^{\alpha_1+1}{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1,\alpha_2\\ \beta_1\end{array};z;b,d\right]\right\}&=\sum_{n=0}^{\infty}(\alpha_1)_n(\alpha_1+n+1)(\alpha_1+n)\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_2+n,\beta_1-\alpha_2)}{B(\alpha_2,\beta_1-\alpha_2)}\frac{z^{n+\alpha_1-1}}{n!}\\&=\alpha_1(\alpha_1+1)z^{\alpha_1}{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1+2,\alpha_2\\ \beta_1\end{array};z;b,d\right]\end{aligned}$$

In the procedure, we have used the fact that $(a+n+1)(a+n)(a)_n=a(a+1)(a+2)_n$. Recursive application of this procedure give us the general form (26). □

A class of recurrence relations for Gauss hypergeometric functions

$${}_2F_1\left[\begin{array}{c}\alpha_1\pm n,\alpha_2\\ \beta_1\end{array};z\right],\ {}_2F_1\left[\begin{array}{c}\alpha_1,\alpha_2\pm n\\ \beta_1\end{array};z\right]\ \text{and}\ {}_2F_1\left[\begin{array}{c}\alpha_1,\alpha_2\\ \beta_1\pm n\end{array};z\right]\quad(n\in\mathbb{N}_0)$$

have been established in [20] as a useful tool to find some new recursion formulas for Appell hypergeometric functions

$$\mathbf{F}_2(\sigma,\alpha_1,\alpha_2\pm n;\beta_1,\beta_2;x,y)\ \text{and}\ \mathbf{F}_2(\sigma,\alpha_1,\alpha_2;\beta_1,\beta_2\pm n;x,y).$$

It is natural to consider whether these results are true for our new definitions. The following theorem gives us a positive answer.

Theorem 2.7. *The following recurrence relations hold true for the extended Gauss hypergeometric functions* ${}_2F_1^{(\kappa_l)}$*:*

1. Recurrence relations for ${}_2F_1^{(\kappa_l)}(\alpha_1\pm n,\alpha_2;\beta_1;z;b,d)$

$${}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1+n,\alpha_2\\ \beta_1\end{array};z;b,d\right]={}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1,\alpha_2\\ \beta_1\end{array};z;b,d\right]+\frac{\alpha_2 z}{\beta_1}\sum_{k=1}^{n}{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1+n-k+1,\alpha_2+1\\ \beta_1+1\end{array};z;b,d\right], \tag{27}$$

$${}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1-n,\alpha_2\\ \beta_1\end{array};z;b,d\right]={}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1,\alpha_2\\ \beta_1\end{array};z;b,d\right]-\frac{\alpha_2 z}{\beta_1}\sum_{k=1}^{n}{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1-k+1,\alpha_2+1\\ \beta_1+1\end{array};z;b,d\right]. \tag{28}$$

$$(|z|<1;\ n\in\mathbb{N}_0)$$

2. Recurrence relation for ${}_2F_1^{(\kappa_l)}(\alpha_1,\alpha_2;\beta_1+n;z;b,d)$

$${}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1,\alpha_2\\ \beta_1+n\end{array};z;b,d\right]=\frac{(\beta_1)_n}{(\beta_1-\alpha_2)_n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\alpha_2)_k}{(\beta_1)_k}{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1,\alpha_2+k\\ \beta_1+k\end{array};z;b,d\right]. \tag{29}$$

$$(|z| < 1;\ n \in \mathbb{N}_0)$$

3. Recurrence relation for ${}_2F_1^{(\kappa_l)}(\alpha_1, \alpha_2 + n; \beta_1; z; b, d)$

$$ {}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1, \alpha_2 + n \\ \beta_1\end{array}; z; b, d\right] = \frac{(\beta_1 - \alpha_2)_{2n}}{(\beta_1 - \alpha_2)_n (\alpha_2)_n} \sum_{i=1}^{n} (-n)_i \frac{(\alpha_2)_{i+n}}{(\beta_1)_{i+n}\, i!}\, {}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1, \alpha_2 + i + n \\ \beta_1 + i + n\end{array}; z; b, d\right]. \tag{30}$$

$$(|z| < 1;\ n \in \mathbb{N}_0)$$

Proof

1. By means of Euler integral representation of the extended Gauss hypergeometric functions given by (13), we have, by writing $(1 - zt)^{-\alpha_1 - n} = (1 - zt)^{-\alpha_1 - n - 1} - zt\,(1 - zt)^{-\alpha_1 - n - 1}$,

$$\begin{aligned} {}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1 + n, \alpha_2 \\ \beta_1\end{array}; z; b, d\right] &= \frac{\Gamma(\beta_1)}{\Gamma(\alpha_2)\Gamma(\beta_1 - \alpha_2)} \int_0^1 t^{\alpha_2 - 1}(1 - t)^{\beta_1 - \alpha_2 - 1}(1 - zt)^{-\alpha_1 - n - 1}\,\Theta\left(\kappa_l; -\frac{b}{t} - \frac{d}{1 - t}\right) dt \\ &\quad - \frac{z\Gamma(\beta_1)}{\Gamma(\alpha_2)\Gamma(\beta_1 - \alpha_2)} \int_0^1 t^{\alpha_2}(1 - t)^{\beta_1 - \alpha_2 - 1}(1 - zt)^{-\alpha_1 - n - 1}\,\Theta\left(\kappa_l; -\frac{b}{t} - \frac{d}{1 - t}\right) dt \\ &= {}_2F_1^{(\kappa_l)}\left(\begin{array}{c}\alpha_1 + n + 1, \alpha_2 \\ \beta_1\end{array}; z; b, d\right) - \left(\frac{z}{\alpha_1 + n}\right)\frac{d}{dz}\left\{{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1 + n, \alpha_2 \\ \beta_1\end{array}; z; b, d\right]\right\}. \end{aligned}$$

Replacing n by $n - 1$, we find that

$$\begin{aligned} {}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1 + n, \alpha_2 \\ \beta_1\end{array}; z; b, d\right] &= {}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1 + n - 1, \alpha_2 \\ \beta_1\end{array}; z; b, d\right] \\ &\quad + \left(\frac{z}{\alpha_1 + n - 1}\right)\frac{d}{dz}\left\{{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1 + n - 1, \alpha_2 \\ \beta_1\end{array}; z; b, d\right]\right\}. \end{aligned} \tag{31}$$

Applying this identity recursively, we obtain:

$$\begin{aligned} {}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1 + n, \alpha_2 \\ \beta_1\end{array}; z; b, d\right] &= {}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1, \alpha_2 \\ \beta_1\end{array}; z; b, d\right] \\ &\quad + z\sum_{k=1}^{n} \frac{1}{\alpha_1 + n - k}\frac{d}{dz}\left\{{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1 + n - k, \alpha_2 \\ \beta_1\end{array}; z; b, d\right]\right\} \\ &= {}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1, \alpha_2 \\ \beta_1\end{array}; z; b, d\right] + z\sum_{k=1}^{n} \frac{\alpha_2}{\beta_1}\,{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1 + n - k + 1, \alpha_2 + 1 \\ \beta_1 + 1\end{array}; z; b, d\right]. \end{aligned} \tag{32}$$

To prove the second formula, we still begin with the integral representation of extended Gauss hypergeometric function. Writing $(1 - zt)^{-(\alpha_1 - n)} = (1 - zt)^{-(\alpha_1 - n + 1)} - zt\,(1 - zt)^{-(\alpha_1 - n + 1)}$ we have

$$\begin{aligned} &{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1 - n, \alpha_2 \\ \beta_1\end{array}; z; b, d\right] = \\ &\quad \frac{\Gamma(\beta_1)}{\Gamma(\alpha_2)\Gamma(\beta_1 - \alpha_2)} \int_0^1 t^{\alpha_2 - 1}(1 - t)^{\beta_1 - \alpha_2 - 1}(1 - zt)^{-(\alpha_1 - n + 1)}\,\Theta\left(\kappa_l; -\frac{b}{t} - \frac{d}{1 - t}\right) dt \\ &\quad - \frac{z\Gamma(\beta_1)}{\Gamma(\alpha_2)\Gamma(\beta_1 - \alpha_2)} \int_0^1 t^{\alpha_2}(1 - t)^{\beta_1 - \alpha_2 - 1}(1 - zt)^{-(\alpha_1 - n + 1)}\,\Theta\left(\kappa_l; -\frac{b}{t} - \frac{d}{1 - t}\right) dt \\ &\quad = {}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1 - n + 1, \alpha_2 \\ \beta_1\end{array}; z; b, d\right] - \left(\frac{z}{\alpha_1 - n}\right)\frac{d}{dz}\left\{{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1 - n, \alpha_2 \\ \beta_1\end{array}; z; b, d\right]\right\}. \end{aligned} \tag{33}$$

Replacing n by $n + 1$, we find that

$$\begin{aligned} {}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1 - n, \alpha_2 \\ \beta_1\end{array}; z; b, d\right] &= {}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1 - n - 1, \alpha_2 \\ \beta_1\end{array}; z; b, d\right] \\ &\quad - z\frac{d}{dz}\left\{\frac{1}{n - \alpha_1 + 1}\,{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1 - n - 1, \alpha_2 \\ \beta_1\end{array}; z; b, d\right]\right\}. \end{aligned} \tag{34}$$

Repeating this recurrence relation n times and applying **Corollary 1**, we obtain:

$$
\begin{aligned}
{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1-n,\alpha_2\\ \beta_1\end{array};z;b,d\right] &= {}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1-2n,\alpha_2\\ \beta_1\end{array};z;b,d\right]\\
&\quad+\frac{\alpha_2 z}{\beta_1}\sum_{k=1}^{n}{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1-n-k+1,\alpha_2+1\\ \beta_1+1\end{array};z;b,d\right]. \qquad (35)
\end{aligned}
$$

The identity (28) follows directly from (35) upon replacing α_1 by α_1+n ($n\in\mathbb{N}_0$).

2. Using Euler integral representation (13) and the fact that

$$
(1-t)^{\beta_1+n-\alpha_2-1}=(1-t)^{\beta_1-\alpha_1-1}(1-t)^n=(1-t)^{\beta_1-\alpha_2-1}\sum_{k=0}^{n}\binom{n}{k}(-1)^k t^k,
$$

we find

$$
\begin{aligned}
{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1,\alpha_2\\ \beta_1+n\end{array};z;b,d\right] &= \frac{\Gamma(\beta_1+n)}{\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_2+n)}\sum_{k=0}^{n}\left(\begin{array}{c}n\\ k\end{array}\right)(-1)^k\int_0^1 t^{\alpha_1+k-1}(1-t)^{\beta_1+k-(\alpha_2+k)-1}\\
&\quad\times(1-zt)^{\alpha-1}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)dt\\
&= \frac{\Gamma(\beta_1+n)}{\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_2+n)}\sum_{k=0}^{n}\left(\begin{array}{c}n\\ k\end{array}\right)(-1)^k\frac{\Gamma(\alpha_2+k)\Gamma(\beta_1-\alpha_2)}{\Gamma(\beta_1+k)}{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1,\alpha_2+k\\ \beta_1+k\end{array};z;b,d\right]\\
&= \frac{(\beta_1)_n}{(\beta_1-\alpha_2)_n}\sum_{k=0}^{n}\left(\begin{array}{c}n\\ k\end{array}\right)(-1)^k\frac{(\alpha_2)_k}{(\beta_1)_k}{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1,\alpha_2+k\\ \beta_1+k\end{array};z;b,d\right].
\end{aligned}
$$

This completes the proof of Eq (29).

3. The demonstration of assertion 3 is based on the following elementary expansion

$$
(1-t)^{-n}=\sum_{i=0}^{\infty}\binom{-n}{i}(-1)^i t^i=\sum_{i=0}^{n}(-n)_i\frac{t^i}{i!}. \qquad (36)
$$

Substituting expansion (36) in the integral representation of extended Gauss hypergeometric function (13) we have

$$
\begin{aligned}
{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1,\alpha_2+n\\ \beta_1\end{array};z;b,d\right] &= \frac{\Gamma(\beta_1)}{\Gamma(\alpha_2+n)\Gamma(\beta_1-\alpha_2-n)}\int_0^1 t^{\alpha_2+n-1}(1-t)^{\beta_1-\alpha_2-n-1}(1-zt)^{-\alpha_1}\\
&\quad\times\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)dt\\
&= \frac{\Gamma(\beta_1)}{\Gamma(\alpha_2)(\alpha_2)_n\Gamma(\beta_1-\alpha_2-n)}\sum_{i=0}^{n}(-n)_i\int_0^1 t^{\alpha_2+n+i-1}(1-t)^{\beta_1-\alpha_2-1}\\
&\quad\times(1-zt)^{-\alpha_1}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)dt\\
&= \frac{\Gamma(\beta_1)}{\Gamma(\alpha_2)(\alpha_2)_n\Gamma(\beta_1-\alpha_2-n)}\sum_{i=0}^{n}(-n)_i\frac{\Gamma(\alpha_2+i+n)\Gamma(\beta_1-\alpha_2)}{\Gamma(\beta_1+i+n)}\\
&\quad\times{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1,\alpha_2+n+i\\ \beta_1+n+i\end{array};z;b,d\right]\\
&= \frac{\Gamma(\beta_1-\alpha_2)}{(\alpha_2)_n\Gamma(\beta_1-\alpha_2-n)}\sum_{i=0}^{n}(-n)_i\frac{(\alpha_2)_{i+n}}{(\beta_1)_{i+n}}{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}\alpha_1,\alpha_2+n+i\\ \beta_1+n+i\end{array};z;b,d\right].
\end{aligned}
$$

And equation (30) follows by using the following property

$$
\frac{\Gamma(\beta_1-\alpha_2)}{\Gamma(\beta_1-\alpha_2-n)}=(\beta_1-\alpha_2+n)_n=\frac{(\beta_1-\alpha_2)_{2n}}{(\beta_1-\alpha_2)_n}.
$$

□

Remark

For classical Gauss hypergeometric function, the parameters α_1 and α_2 are symmetric, i.e.,

$$ {}_2F_1(\alpha_1,\alpha_2;\beta_1;z) = {}_2F_1(\alpha_2,\alpha_1;\beta_1;z). $$

Thus we only need to establish the recurrence relations for one of them and the results can be valid for both α_1 and α_2. However, the numerator parameters for extended Gauss hypergeometric functions possess no such symmetry. ♢

2.3. ***A further generalization and its connection with fractional calculus***

Our definition of extended generalized hypergeometric function can in fact be generalized to the following form:

Definition 2.2. *For suitably constrained (real or complex) parameters* $\alpha_j,\ j=1,\cdots,p$*;* $\beta_i,\ i=1,\cdots,q$*, we define the extended generalized hypergeometric functions by*

$$ {}_pF_q^{(\kappa_l)}\left[\begin{array}{ccc}(\alpha_1,k_1) & \cdots & (\alpha_p,k_p)\\ \beta_1, & \cdots & \beta_q,\end{array};z;b,d\right] = $$
$$ \begin{cases} \displaystyle\sum_{m=0}^{\infty}(\alpha_1)_{k_1 m}\prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_{j+1}+k_{j+1}m,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1},\beta_j-\alpha_{j+1})}\frac{z^m}{m!}, & |z|<1;\ p=q+1\\ \displaystyle\sum_{m=0}^{\infty}\prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_j+k_jm,\beta_j-\alpha_j)}{B(\alpha_j,\beta_j-\alpha_j)}\frac{z^m}{m!}, & p=q\\ \displaystyle\sum_{m=0}^{\infty}\frac{1}{(\beta_1)_m}\cdots\frac{1}{(\beta_r)_m}\prod_{j=1}^{p}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_j+k_jm,\beta_{r+j}-\alpha_j)}{B(\alpha_j,\beta_{r+j}-\alpha_j)}\frac{z^m}{m!}, & r=q-p,\ p<q\end{cases} \tag{37} $$

where the new introduced parameters $k_j,\ j=1,\cdots,p$ *are non-negative integers.*

Obviously, (37) reduces to (18) whenever $k_j=1,\ j=1,\cdots,p$. To illustrate its advantages we first consider the following function:

$$ {}_2F_1^{(\kappa_l)}\left[\begin{array}{c}(\alpha_1,1),(\alpha_2,k_2)\\ \beta_1\end{array};z;b,d\right] = \sum_{n=0}^{\infty}(\alpha_1)_n\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_2+k_2n,\beta_1-\alpha_2)}{B(\alpha_2,\beta_1-\alpha_2)}\frac{z^n}{n!}. \tag{38} $$

Its integral representation can be written as

$$ {}_2F_1^{(\kappa_l)}\left[\begin{array}{c}(\alpha_1,1),(\alpha_2,k_2)\\ \beta_1\end{array};z;b,d\right] = \frac{1}{B(\alpha_2,\beta_1-\alpha_2)} $$
$$ \int_0^1 t^{\alpha_2-1}(1-t)^{\beta_1-\alpha_2-1}\left(1-zt^{k_2}\right)^{-\alpha_1}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)dt. \tag{39} $$

$$ \Re(\beta_1)>\Re(\alpha_2)>0;\Re(b),\Re(d)>0;b=d=0,|\arg(1-z)<\pi| $$

As an example, let us compute the case for $k_1 = 2$ and $z = 1$,

$$\begin{aligned}
&\frac{\Gamma(\beta_1)}{\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_2)}\int_0^1 t^{\alpha_2-1}(1-t)^{\beta_1-\alpha_2-1}\left(1-t^2\right)^{-\alpha_1}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)dt\\
&=\frac{\Gamma(\beta_1)}{\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_2)}\int_0^1 t^{\alpha_2-1}(1-t)^{\beta_1-\alpha_2-1}(1-t)^{-\alpha_1}(1+t)^{-\alpha_1}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)dt\\
&=\frac{\Gamma(\beta_1)}{\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_2)}\sum_{k=0}^{\infty}\binom{-\alpha_1}{k}\int_0^1 t^{\alpha_2+k-1}(1-t)^{\beta_1-\alpha_2-\alpha_1-1}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)dt\\
&=\frac{\Gamma(\beta_1)}{\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_2)}\sum_{k=0}^{\infty}\frac{(-1)^k(\alpha_1)_k}{k!}\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_2+k,\beta_1-\alpha_2-\alpha_1)\\
&=\frac{\Gamma(\beta_1)}{\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_2)}\frac{\Gamma(\beta_1-\alpha_2-\alpha_1)\Gamma(\alpha_2)}{\Gamma(\beta_1-\alpha_1)}\sum_{k=0}^{\infty}(\alpha_1)_k\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_2+k,\beta_1-\alpha_2-\alpha_1)}{B(\alpha_2,\beta_1-\alpha_2-\alpha_1)}\frac{(-1)^k}{k!}\\
&=\frac{\Gamma(\beta_1)\Gamma(\beta_1-\alpha_2-\alpha_1)}{\Gamma(\beta_1-\alpha_2)\Gamma(\beta_1-\alpha_1)}{}_2F_1^{(\kappa_l)}\left[\begin{matrix}\alpha_1,\alpha_2\\ \beta_1-\alpha_1\end{matrix};-1;b,d\right].
\end{aligned}$$

We conclude our result as the following theorem:

Theorem 2.8. *If* $\Re(\beta_1) > \Re(\alpha_2) > 0$ *and* $\Re(\beta_1-\alpha_2-\alpha_1) > 0$, *then*

$$ {}_2F_1^{(\kappa_l)}\left[\begin{matrix}(\alpha_1,1),(\alpha_2,2)\\ \beta_1\end{matrix};1;b,d\right]=\frac{\Gamma(\beta_1)\Gamma(\beta_1-\alpha_2-\alpha_1)}{\Gamma(\beta_1-\alpha_2)\Gamma(\beta_1-\alpha_1)}{}_2F_1^{(\kappa_l)}\left[\begin{matrix}\alpha_1,\alpha_2\\ \beta_1-\alpha_1\end{matrix};-1;b,d\right]. \tag{40}$$

Theorem 2.8 is in fact a generalization of the following well-known result [9, p.117, Theorem 2.3]:

$$ {}_3F_2\left[\begin{matrix}a,\frac{b}{2},\frac{b+1}{2}\\ \frac{c}{2},\frac{c+1}{2}\end{matrix};1\right]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}{}_2F_1\left[\begin{matrix}a,b\\ c-a\end{matrix};-1\right]. \tag{41}$$

It also show some intrinsic relationships between our extended hypergeometric function and the ordinary generalized hypergeometric functions.

In general, we have for definition (37) the following integral representation, which is a generalization of [23, p.104, Theorem 38].

Theorem 2.9. *If* $p \le q+1$, $\Re(\beta_q) > \Re(\alpha_p) > 0$, $k_j \in \mathbb{Z}^+\cup\{0\}, j=1,\cdots,p$ *and none of* $\beta_1,\ldots,\beta_q$ *is zero or a negative integer, we have (inside the region of convergence)*

$$\begin{aligned}
{}_pF_q^{(\kappa_l)}\left[\begin{matrix}(\alpha_1,k_1)&\cdots&(\alpha_p,k_p)\\ \beta_1&\cdots&\beta_q\end{matrix};cz^{k_p};b,d\right]&=\frac{\Gamma(\beta_q)z^{1-\beta_q}}{\Gamma(\alpha_p)\Gamma(\beta_q-\alpha_p)}\int_0^z t^{\alpha_p-1}(z-t)^{\beta_q-a_p-1}\\
&\times{}_{p-1}F_{q-1}^{(\kappa_l)}\left[\begin{matrix}(\alpha_1,k_1)&\cdots&(\alpha_{p-1},k_{p-1})\\ \beta_1&\cdots&\beta_{q-1}\end{matrix};ct^{k_p};b,d\right]\\
&\times\Theta\left(\kappa_l;-\frac{bz}{t}-\frac{dz}{z-t}\right)dt.
\end{aligned}\tag{42}$$

Proof

Denote the right-hand side of (42) by $A(z)$. As indicated before, we just need to prove the case that $p = q + 1$. Let $t = zv$ in $A(z)$. Then we have

$$
\begin{aligned}
A(z) &= \frac{\Gamma(\beta_q) z^{\beta_q - 1}}{\Gamma(\alpha_p)\Gamma(\beta_q - \alpha_p)} \int_0^1 v^{\alpha_{q+1}-1} (1-v)^{\beta_q - a_{q+1} - 1} \\
&\times {}_qF_{q-1}^{(\kappa_l)} \left[\begin{matrix} (\alpha_1, k_1) & \cdots & (\alpha_q, k_q) \\ \beta_1 & \cdots & \beta_{q-1} \end{matrix} ; c(zv)^{k_{q+1}}; b, d \right] \Theta\left(\kappa_l; -\frac{b}{v} - \frac{d}{1-v}\right) dv \\
&= \frac{\Gamma(\beta_q) z^{\beta_q - 1}}{\Gamma(\alpha_{q+1})\Gamma(\beta_q - \alpha_{q+1})} \sum_{m=0}^{\infty} (\alpha_1)_{k_1 m} \prod_{j=1}^{q-1} \frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_{j+1} + k_{j+1} m, \beta_j - \alpha_{j+1})}{B(\alpha_{j+1}, \beta_j - \alpha_{j+1})} \frac{(cz^{k_{q+1}})^m}{m!} \\
&\times \int_0^1 v^{\alpha_{q+1} + k_{q+1} m - 1} (1-v)^{\beta_q - a_{q+1} - 1} \Theta\left(\kappa_l; -\frac{b}{v} - \frac{d}{1-v}\right) dv \\
&= z^{\beta_q - 1} \sum_{m=0}^{\infty} (\alpha_1)_{k_1 m} \prod_{j=1}^{q} \frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_{j+1} + k_{j+1} m, \beta_j - \alpha_{j+1})}{B(\alpha_{j+1}, \beta_j - \alpha_{j+1})} \frac{(cz^{k_{q+1}})^m}{m!} \\
&= z^{\beta_q - 1} {}_{q+1}F_q^{(\kappa_l)} \left[\begin{matrix} (\alpha_1, k_1) & \cdots & (\alpha_q, k_q) & (\alpha_{q+1}, k_{q+1}) \\ \beta_1 & \cdots & \beta_{q-1} & \beta_q \end{matrix} ; cz^{k_{q+1}}; b, d \right].
\end{aligned}
$$

□

Theorem 2.9 embody the significance and advantages of our modified definition (37) of extended generalized hypergeometric function. It not only enables us to rebuild a lot of classical results of hypergeometric function in a new way but also provide some new important interpretations. Here is one of them. In [18], H. M. Srivastava definite the following fractional derivative operator:

$$
\mathcal{D}_{z,(\kappa_l)}^{\mu,b,d}\{f(z)\} = \begin{cases} \frac{1}{\Gamma(-\mu)} \int_0^z (z-t)^{-\mu-1} \Theta\left(\kappa_l; -\frac{bz}{t} - \frac{dz}{z-t}\right) f(t)\, dt & (\Re(\mu) < 0) \\ \frac{d^m}{dz^m} \left\{ \mathcal{D}_{z,(\kappa_l)}^{\mu-m,b,d} \{f(z)\} \right\} & (m - 1 \le \Re(\mu) \le m\ (m \in \mathbb{N})) \end{cases} \tag{43}
$$

The path of integration in (43) is a line in the complex t−plane from $t = 0$ to $t = z$.

By using operator (43) we can rewrite (42) as

$$
\begin{aligned}
&{}_pF_q^{(\kappa_l)} \left[\begin{matrix} (\alpha_1, k_1) & \cdots & (\alpha_p, k_p) \\ \beta_1 & \cdots & \beta_q \end{matrix} ; cz^{k_p}; b, d \right] \\
&= \frac{\Gamma(\beta_q)}{\Gamma(\alpha_p)} z^{1-\beta_q} \mathcal{D}_{z,(\kappa_l)}^{-(\beta_q - \alpha_p),b,d} \left\{ t^{\alpha_p - 1} {}_{p-1}F_{q-1}^{(\kappa_l)} \left[\begin{matrix} (\alpha_1, k_1) & \cdots & (\alpha_{p-1}, k_{p-1}) \\ \beta_1 & \cdots & \beta_{q-1} \end{matrix} ; ct^{k_p}; b, d \right] \right\}.
\end{aligned} \tag{44}
$$

3. Extended Appell and Lauricella Hypergeometric Functions

In this section, we focus on some extended multivariate hypergeometric functions, i.e., extended Appell's hypergeometric functions and extended Lauricella's hypergeometric functions.

For the sake of clarity and easy readability, we may first study the properties of extended Appell's hypergeometric functions and then we can view the extended Lauricella's hypergeometric functions as a further generalization of Appell functions.

Definition 3.1. *[18]: The extended Appell's hypergeometric functions*

$$
\mathbf{F}_1^{(\kappa_l)}(\alpha, \beta_1, \beta_2; \gamma_1; x, y; b, d) \text{ and } \mathbf{F}_2^{(\kappa_l)}(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y; b, d)
$$

in two variables are defined by

$$
\mathbf{F}_1^{(\kappa_l)}(\alpha, \beta_1, \beta_2; \gamma_1; x, y; b, d) = \sum_{m,n=0}^{\infty} (\beta_1)_m (\beta_2)_n \frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha + m + n, \gamma_1 - \alpha)}{B(\alpha, \gamma_1 - \alpha)} \frac{x^m y^n}{m!n!}, \tag{45}
$$

$$
(\max\{|x|, |y|\} < 1; \min\{\Re(b), \Re(d)\} \ge 0)
$$

$$\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y;b,d)=\sum_{m,n=0}^{\infty}(\alpha)_{m+n}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\beta_1+m,\gamma_1-\beta_1)\,\mathrm{B}_{b,d}^{(\kappa_l)}(\beta_2+n,\gamma_2-\beta_2)}{\mathrm{B}(\beta_1,\gamma_1-\beta_1)\,\mathrm{B}(\beta_2,\gamma_2-\beta_2)}\frac{x^m y^n}{m!n!}. \tag{46}$$

$$(|x|+|y|<1;\min\{\Re(b),\Re(d)\}\geq 0)$$

The following Euler type integral representations for Extended Appell hypergeometric functions $\mathbf{F}_1^{(\kappa_l)}$ and $\mathbf{F}_2^{(\kappa_l)}$ are easily obtained and their proofs will be omitted.

Theorem 3.1. *For the extended Appell functions defined by (45) and (46), the following integral representations hold true:*

$$\mathbf{F}_1^{(\kappa_l)}(\alpha,\beta_1,\beta_2;\gamma_1;x,y;b,d)$$
$$=\frac{1}{\mathrm{B}(\alpha,\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma_1-\alpha-1}(1-xt)^{-\beta_1}(1-yt)^{-\beta_2}\,\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)dt, \tag{47}$$

$$(\min\{\Re(b),\Re(d)\}\geq 0;\max\{|\arg(1-x)|,|\arg(1-y)|\}<\pi;\Re(\gamma)>\Re(\alpha)>0)$$

$$\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y;b,d)=\frac{1}{\mathrm{B}(\beta_1,\gamma_1-\beta_1)}\frac{1}{\mathrm{B}(\beta_2,\gamma_2-\beta_2)}$$
$$\int_0^1\int_0^1\frac{t^{\beta_1-1}(1-t)^{\gamma_1-\beta_1-1}s^{\beta_2-1}(1-s)^{\gamma_2-\beta_2-1}}{(1-xt-ys)^{\alpha}}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)\Theta\left(\kappa_l;-\frac{b}{s}-\frac{d}{1-s}\right)dtds. \tag{48}$$

$$(\min\{\Re(b),\Re(d)\}\geq 0;|x|+|y|<1;\Re(\gamma_1)>\Re(\beta_1)>0,\Re(\gamma_2)>\Re(\beta_2)>0)$$

By using the integral representations of $\mathbf{F}_1^{(\kappa_l)}(\alpha,\beta,\beta';\gamma;x,y;b,d)$ and $\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta,\beta';\gamma,\gamma';x,y;b,d)$, we can derive some transformation formulas for them.

Theorem 3.2. *For the extended Appell hypergeometric function $\mathbf{F}_1^{(\kappa_l)}$, we have the following transformation formula:*

$$\mathbf{F}_1^{(\kappa_l)}(\alpha,\beta,\beta';\gamma;x,y;b,d)=(1-x)^{-\beta}(1-y)^{-\beta'}\mathbf{F}_1^{(\kappa_l)}\left(\alpha,\beta,\beta';\gamma;\frac{x}{x-1},\frac{y}{y-1};d,b\right). \tag{49}$$

Proof

Let us put $t=1-s$. Then

$$\begin{aligned}\mathbf{F}_1^{(\kappa_l)}(\alpha,\beta,\beta';\gamma;x,y;b,d)&=\frac{1}{\mathrm{B}(\alpha,\gamma-\alpha)}\int_0^1 s^{\gamma-\alpha-1}(1-s)^{\alpha-1}(1-x(1-s))^{-\beta}(1-y(1-s))^{-\beta'}\\&\times\Theta\left(\kappa_l;-\frac{b}{1-s}-\frac{d}{s}\right)ds\\&=\frac{1}{\mathrm{B}(\alpha,\gamma-\alpha)}\int_0^1 s^{\gamma-\alpha-1}(1-s)^{\alpha-1}(1-x+xs)^{-\beta}(1-y+ys)^{-\beta'}\\&\times\Theta\left(\kappa_l;-\frac{b}{1-s}-\frac{d}{s}\right)ds\\&=\frac{(1-x)^{-\beta}(1-y)^{\beta'}}{\mathrm{B}(\alpha,\gamma-\alpha)}\int_0^1 s^{\gamma-\alpha-1}(1-s)^{\alpha-1}\left(1-\frac{x}{x-1}s\right)^{-\beta}\left(1-\frac{y}{y-1}s\right)^{-\beta'}\\&\times\Theta\left(\kappa_l;-\frac{d}{s}-\frac{b}{1-s}\right)ds\end{aligned}$$

such that

$$\mathbf{F}_1^{(\kappa_l)}(\alpha,\beta,\beta';\gamma;x,y;b,d)=(1-x)^{-\beta}(1-y)^{\beta'}\mathbf{F}_1^{(\kappa_l)}\left(\gamma-\alpha,\beta,\beta';\gamma;\frac{x}{x-1},\frac{y}{y-1};d,b\right).$$

$\square$

Remark

The classical theory of Appell hypergeometric functions have found that integral

$$\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}\,du$$

keeps unchanged under the following five transformations [24, p.195]

$$\left.\begin{array}{ll} u=1-v, & \\ u=\frac{v}{1-y+vy}, & u=\frac{1-v}{1-vx}, \\ u=\frac{v}{1-x+vx}, & y=\frac{1-v}{1-vy}. \end{array}\right\}$$

However, only the first transformation $u = 1 - v$ enable us to get a simple extension. ♢

Theorem 3.3. *For* $b = d$ *we have the following transformation formulas for Appell hypergeometric function* $\mathbf{F}_2^{(\kappa_l)}$*:*

$$\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta,\beta';\gamma,\gamma';x,y;b) = (1-x)^{-\alpha}\,\mathbf{F}_2^{(\kappa_l)}\left(\alpha,\gamma-\beta,\beta';\gamma,\gamma';\frac{x}{x-1},\frac{-y}{x-1};b\right), \tag{50}$$

$$\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta,\beta';\gamma,\gamma';x,y;b) = (1-y)^{-\alpha}\,\mathbf{F}_2^{(\kappa_l)}\left(\alpha,\beta,\gamma'-\beta';\gamma,\gamma';\frac{-x}{y-1},\frac{y}{y-1};b\right), \tag{51}$$

$$\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta,\beta';\gamma,\gamma';x,y;b) = (1-x-y)^{-\alpha}\,\mathbf{F}_2^{(\kappa_l)}\left(\alpha,\gamma-\beta,\gamma'-\beta';\gamma,\gamma';\frac{-x}{1-x-y},\frac{-y}{1-x-y};b\right). \tag{52}$$

Proof

By using the double integral representation of $\mathbf{F}_2^{(\kappa_l)}$ with $b = d$ and the transformations

$$\left.\begin{array}{lcc} (a) & t=1-t', & s=s'; \\ (b) & t=t' & s=s-1; \\ (c) & t=1-t', & s=1-s'; \end{array}\right\}$$

we can directly obtain the transformation formulas for $\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta,\beta';\gamma,\gamma';x,y;b)$, respectively.

Note that formula (52) is also valid for more general case, namely,

$$\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta,\beta';\gamma,\gamma';x,y;b,d) = (1-x-y)^{-\alpha}\,\mathbf{F}_2^{(\kappa_l)}\left(\alpha,\gamma-\beta,\gamma'-\beta';\gamma,\gamma';\frac{-x}{1-x-y},\frac{-y}{1-x-y};d,b\right). \tag{53}$$

□

3.1. ***Other Integral Representations of Appell's Hypergeometric Functions*** $\mathbf{F}_1^{(\kappa_l)}$ ***and*** $\mathbf{F}_2^{(\kappa_l)}$

We now start to establish the Mellin-Barnes type integral for Appell functions $\mathbf{F}_1^{(\kappa_l)}$ and $\mathbf{F}_2^{(\kappa_l)}$. The tool we apply here is called *Method of Bracket* [1], which can be view as a multidimensional extension to Ramanujan's Master Theorem. This method enable us to evaluate multidimensional Mellin transform directly and efficiently. In this part, we will prove the following theorem with this method. Here we will put focus on the proving instead of explaining the method in detail. Because the method is easy-understanding, we will give a brief introduction of it when proving **Theorem 3.4.** More detailed information may be found in [1] and [11].

Theorem 3.4. *For suitable parameters, the Mellin-Barnes integral representation of* $\mathbf{F}_1^{(\kappa_l)}$ *is*

$$\begin{aligned}\mathbf{F}_1^{(\kappa_l)}(\alpha,\beta_1,\beta_2;\gamma_1;x,y;b,d) = {} & \frac{1}{(2\pi i)^2}\frac{\Gamma(\gamma_1)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha)\Gamma(\gamma_1-\alpha)}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha-s_1-s_2,\gamma_1-\alpha) \\ & \times\Gamma(\beta_1-s_1)\Gamma(\beta_2-s_2)\Gamma(s_1)\Gamma(s_2)(-x)^{-s_1}(-y)^{-s_2}\,ds_1ds_2. \end{aligned} \tag{54}$$

Proof

We first take the Mellin transform of $\mathbf{F}_1^{(\kappa_l)}(\alpha,\beta_1,\beta_2;\gamma_1;-x,-y;b,d)$, i.e.,

$$\int_0^\infty\int_0^\infty x^{s_1-1}y^{s_2-1}\mathbf{F}_1^{(\kappa_l)}(\alpha,\beta_1,\beta_2;\gamma_1;-x,-y;b,d)\,dxdy=\sum_{m,n=0}^{\infty}(\beta_1)_m(\beta_2)_n\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha+m+n,\gamma_1-\alpha)}{B(\alpha,\gamma_1-\alpha)}$$
$$\times\frac{(-1)^m}{\Gamma(m+1)}\frac{(-1)^n}{\Gamma(n+1)}\int_0^\infty x^{s_1+m-1}dx\int_0^\infty y^{s_2+n-1}dy=\sum_{m,n=0}^{\infty}\phi_m\phi_n f(m,n)\langle s_1+m\rangle\langle s_2+n\rangle \quad (55)$$

where

$$f(m,n)=(\beta_1)_m(\beta_2)_n\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha+m+n,\gamma_1-\alpha)}{B(\alpha,\gamma_1-\alpha)}=\frac{\Gamma(\beta_1+m)}{\Gamma(\beta_1)}\frac{\Gamma(\beta_2+n)}{\Gamma(\beta_2)}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha+m+n,\gamma_1-\alpha)}{B(\alpha,\gamma_1-\alpha)},$$
$$\phi_m=\frac{(-1)^m}{\Gamma(m+1)},\ \phi_n=\frac{(-1)^n}{\Gamma(n+1)},$$
$$\langle s_1+m\rangle=\int_0^\infty x^{s_1+m-1}dx,\ \langle s_2+n\rangle=\int_0^\infty y^{s_2+n-1}dy.$$

The symbol ϕ_m and ϕ_n are called the *indicator* of m and n, respectively [11, p.53, Definition 2.2]. Then use the *method of brackets* **Rule III** [1] we have

$$\int_0^\infty\int_0^\infty x^{s_1-1}y^{s_2-1}\mathbf{F}_1^{(\kappa_l)}(\alpha,\beta_1,\beta_2;\gamma_1;-x,-y;b,d)\,dxdy=\frac{1}{|\det(A)|}\Gamma(-m^*)\Gamma(-n^*)f(m^*,n^*)$$
$$=\frac{\Gamma(\beta_1-s_1)}{\Gamma(\beta_1)}\frac{\Gamma(\beta_2-s_2)}{\Gamma(\beta_2)}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha-s_1-s_2,\gamma_1-\alpha)}{B(\alpha,\gamma_1-\alpha)}\Gamma(s_1)\Gamma(s_2), \quad (56)$$

where (m^*,n^*) is obtained by vanishing the expressions in the brackets $\langle s_1+m\rangle$, $\langle s_2+n\rangle$. In general, (m^*,n^*) is the unique solution to the linear system obtained by vanishing the expressions in the brackets. And the determinant of its coefficient matrix is denoted by $\det(A)$. Here, we can see $\langle a\rangle$ is only a symbol associated with divergent integrals $\int_0^\infty x^{a-1}dx$. But by introducing such a symbol, we in fact reduce the evaluation of definite integral to solving a linear system of equations, which is the basic idea of the *method of brackets*.

Now we can take the inverse Mellin transform of (56) to get

$$\mathbf{F}_1^{(\kappa_l)}(\alpha,\beta_1,\beta_2;\gamma_1;-x,-y;b,d)=\frac{1}{(2\pi i)^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\frac{\Gamma(\beta_1-s_1)}{\Gamma(\beta_1)}\frac{\Gamma(\beta_2-s_2)}{\Gamma(\beta_2)}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha-s_1-s_2,\gamma_1-\alpha)}{B(\alpha,\gamma_1-\alpha)}$$
$$\times\Gamma(s_1)\Gamma(s_2)x^{-s_1}y^{-s_2}ds_1ds_2.$$

Theorem 3.4 follows from replacing $-x$ and $-y$ with x and y. □

The Mellin-barnes double contour integral representation for extended Appell's hypergeometric function $\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y;b,d)$ is given in the following theorem. We prove this just the same way when proving **Theorem 3.4**.

Theorem 3.5. *For suitable parameters, the Mellin-Barnes integral representation of* $\mathbf{F}_2^{(\kappa_l)}$ *is*

$$\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y;b,d)=\frac{1}{(2\pi i)^2}\frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\beta_1)\Gamma(\gamma_1-\beta_1)\Gamma(\beta_2)\Gamma(\gamma_2-\beta_2)\Gamma(\alpha)}$$
$$\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\mathcal{B}_{b,d}^{(\kappa_l)}(\beta_1-s_1,\gamma_1-\beta_1)B_{b,d}^{(\kappa_l)}(\beta_2-s_2,\gamma_2-\beta_2)\Gamma(\alpha-s_1-s_2)\Gamma(s_1)\Gamma(s_2)(-x)^{-s_1}(-y)^{-s_2}ds_1ds_2.$$

3.2. ***Recursion formulas for Appell hypergeometric function*** $\mathbf{F}_2^{(\kappa_l)}$

Recursion formulas for classical Appell hypergeometric function $\mathbf{F}_2$ have been studied in [20] by using contiguous function relations of the classical Gauss hypergeometric series ${}_2F_1$. These recursion formulas can be used to evaluate the radiation field integrals. Now we try to establish the similar formulas for our new Appell hypergeometric function $\mathbf{F}_2^{(\kappa_l)}$.

Theorem 3.6. *The extended Appell hypergeometric function* $\mathbf{F}_2^{(\kappa_l)}$ *satisfies the following identities:*

$$\begin{aligned}&\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1,\beta_2+n;\gamma_1,\gamma_2;x,y;b,d)\\&\quad=\frac{(\gamma_2-\beta_2)_{2n}}{(\gamma_2-\beta_2)_n(\beta_2)_n}\sum_{i=1}^{n}(-n)_i\frac{(\beta_2)_{i+n}}{(\gamma_2)_{i+n}\,i!}\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1,\beta_2+n+i;\gamma_1,\gamma_2+i+n;x,y;b,d)\end{aligned}\tag{57}$$

$$\left(|x|+|y|<1;\min\{\Re(b),\Re(d)\}\geq 0;\ n\in\mathbb{N}_0,\alpha,\beta_1,\beta_2\in\mathbb{C};\gamma_1,\gamma_2\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

and

$$\begin{aligned}&\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2+n;x,y;b,d)\\&\quad=\frac{(\gamma_2)_n}{(\gamma_2-\beta_2)_n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\beta_2)_k}{(\gamma_2)_k}\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1,\beta_2+k;\gamma_1,\gamma_2+k;x,y;b,d).\end{aligned}\tag{58}$$

$$\left(|x|+|y|<1;\min\{\Re(b),\Re(d)\}\geq 0;\ n\in\mathbb{N}_0,\alpha,\beta_1,\beta_2\in\mathbb{C};\gamma_1,\gamma_2\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

Proof

1. Using the integral representation (13) the following double integral representation

$$\begin{aligned}&\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y;b,d)=\frac{1}{B(\beta_1,\gamma_1-\beta_1)}\frac{1}{B(\beta_2,\gamma_2-\beta_2)}\\&\times\int_0^1\int_0^1\frac{t^{\beta_1-1}(1-t)^{\gamma_1-\beta_1-1}s^{\beta_2-1}(1-s)^{\gamma_2-\beta_2-1}}{(1-xt-ys)^{\alpha}}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)\Theta\left(\kappa_l;-\frac{b}{s}-\frac{d}{1-s}\right)dtds\end{aligned}$$

can be reduced to a single integral involving the extended Gauss hypergeometric function ${}_2F_1^{(\kappa_l)}$, i.e.,

$$\begin{aligned}\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y;b,d)&=\frac{1}{B(\beta_1,\gamma_1-\beta_1)}\int_0^1 t^{\beta_1-1}(1-t)^{\gamma_1-\beta_1-1}(1-xt)^{-\alpha}\\&\quad{}_2F_1^{(\kappa_l)}\left(\begin{array}{c}\alpha,\beta_2\\ \gamma_2\end{array};\frac{y}{1-xt};b,d\right)\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)dt.\end{aligned}\tag{59}$$

The assertion (57) then follows by substituting (30) into integral representation (59) with $\beta_2=\beta_2+n$. Thus

$$\begin{aligned}&\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1,\beta_2+n;\gamma_1,\gamma_2;x,y;b,d)\\&=\frac{1}{B(\beta_1,\gamma_1-\beta_1)}\int_0^1 t^{\beta_1-1}(1-t)^{\gamma_1-\beta_1-1}(1-xt)^{-\alpha}{}_2F_1^{(\kappa_l)}\left(\begin{array}{c}\alpha,\beta_2+n\\ \gamma_2\end{array};\frac{y}{1-xt};b,d\right)\\&\times\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)dt\end{aligned}\tag{60}$$

$$\begin{aligned}&=\frac{1}{B(\beta_1,\gamma_1-\beta_1)}\frac{(\gamma_2-\beta_2)_{2n}}{(\gamma_2-\beta_2)_n(\beta_2)_n}\sum_{i=1}^{n}(-n)_i\frac{(\beta_2)_{i+n}}{(\gamma_2)_{i+n}\,i!}\int_0^1 t^{\beta_1-1}(1-t)^{\gamma_1-\beta_1-1}(1-xt)^{-\alpha}\\&\times{}_2F_1^{(\kappa_l)}\left(\begin{array}{c}\alpha,\beta_2+i+n\\ \gamma_2+i+n\end{array};\frac{y}{1-xt};b,d\right)\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)dt\end{aligned}\tag{61}$$

$$=\frac{(\gamma_2-\beta_2)_{2n}}{(\gamma_2-\beta_2)_n(\beta_2)_n}\sum_{i=1}^{n}(-n)_i\frac{(\beta_2)_{i+n}}{(\gamma_2)_{i+n}\,i!}\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1,\beta_2+n+i;\gamma_1,\gamma_2+i+n;x,y;b,d).\tag{62}$$

2. The proof of the second assertion of **Theorem 3.6** follows directly by substituting from the the assertion (29) of **Theorem 2.7** into the integral representation (59) with $\gamma_2 = \gamma_2 + n$,

$$\begin{aligned}
&\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2+n;x,y;b,d)\\
&= \frac{1}{B(\beta_1,\gamma_1-\beta_1)}\int_0^1 t^{\beta_1-1}(1-t)^{\gamma_1-\beta_1-1}(1-xt)^{-\alpha}\,{}_2F_1^{(\kappa_l)}\left(\begin{array}{c}\alpha,\beta_2\\ \gamma_2+n\end{array};\frac{y}{1-xt};b,d\right)\\
&\quad\times\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)dt\\
&= \frac{(\gamma_2)_n}{(\gamma_2-\beta_2)_n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\beta_2)_k}{(\gamma_2)_k}\int_0^1\frac{t^{\beta_1-1}(1-t)^{\gamma_1-\beta_1-1}}{B(\beta_1,\gamma_1-\beta_1)}(1-xt)^{-\alpha}\\
&\quad\times{}_2F_1^{(\kappa_l)}\left(\begin{array}{c}\alpha,\beta_2+k\\ \gamma_2+k\end{array};\frac{y}{1-xt};b,d\right)\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)dt \qquad (63)\\
&= \frac{(\gamma_2)_n}{(\gamma_2-\beta_2)_n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\beta_2)_k}{(\gamma_2)_k}\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1,\beta_2+k;\gamma_1,\gamma_2+k;x,y;b,d).
\end{aligned}$$

□

Remark

The second assertion (58) of **Theorem 3.6** is clear a extension of the formula given in [20, p.549, Theorem 3] and the first formula (57) seem to be new. In addition, the results obtained here are also applicable to the extended Appell hypergeometric functions

$$\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1,\beta_2;\gamma_1+n,\gamma_2;x,y;b,d) \text{ and } \mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1+n,\beta_2;\gamma_1,\gamma_2;x,y;b,d).$$

It seems to be a little difficult to establish some recursion relations for the functions:

$$\mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1,\beta_2;\gamma_1-n,\gamma_2;x,y;b,d) \text{ and } \mathbf{F}_2^{(\kappa_l)}(\alpha,\beta_1-n,\beta_2;\gamma_1,\gamma_2;x,y;b,d).$$

♢

3.3. ***Finite sum representation of the extended Appell's hypergeometric function*** $\mathbf{F}_1^{(\kappa_l)}$

Recently, Cuyt et al. [8] obtained a finite algebraic sum representation of the Appell hypergeometric function $\mathbf{F}_1$ in the case when the parameters α, β_1, β_2 and $\gamma_1$are positive integers with $\gamma_1 > \alpha$. Motivated by their works, we show that their results can be extended to obtain a finite sum representation of extended Appell's hypergeometric function $\mathbf{F}_1^{(\kappa_l)}$.

Before we state our results, we need the following important lemma.

Lemma 1. *[8]: Let* $\alpha = \frac{y}{y-x}$, $\beta = \frac{x}{x-y}$ *and* $X = (1-ux)^{-1}$, $Y = (1-uy)^{-1}$. *Then for any positive integers* s *and* t,

$$X^sY^t = \alpha^s\sum_{j=0}^{t-1}\binom{j+s-1}{s-1}\beta^jY^{t-j} + \beta^t\sum_{k=0}^{s-1}\binom{k+t-1}{t-1}\alpha^kX^{s-k}. \qquad (64)$$

Theorem 3.7. *For any non-negative integers* s, t *and* $x \neq y$, *we have for* $|x| < 1$, $|y| < 1$

$$\begin{aligned}
\mathbf{F}_1^{(\kappa_l)}(1,s+1,t+1;2;x,y;b,d) &= y^{s+1}\sum_{j=0}^{t-1}\binom{j+s}{s}\frac{(-x)^j}{(y-x)^{j+s+1}}{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}(t-j+1),1\\ 2\end{array};y;b,d\right]\\
&+ x^{t+1}\sum_{k=0}^{s-1}\binom{k+t}{t}\frac{(-y)^k}{(y-x)^{k+t+1}}{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}(s-k+1),1\\ 2\end{array};x;b,d\right]\\
&+ \binom{t+s}{s}\frac{(-1)^t x^t y^s}{(y-x)^{t+s+1}}\left\{y\,{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}1,1\\ 2\end{array};y;b,d\right] + x\,{}_2F_1^{(\kappa_l)}\left[\begin{array}{c}1,1\\ 2\end{array};x;b,d\right]\right\}.
\end{aligned} \qquad (65)$$

Proof

Put $\alpha = 1$, $\beta_1 = s+1$, $\beta_2 = t+1$ and $\gamma_2 = 2$ in the integral representation (47), we obtain

$$\mathbf{F}_1^{(\kappa_l)}(1, s+1, t+1; 2; x, y; b, d) = \int_0^1 (1-xu)^{-(s+1)}(1-yu)^{-(t+1)}\,\Theta\left(\kappa_l; -\frac{b}{u} - \frac{d}{1-u}\right) du. \tag{66}$$

By using **lemma 1** we have

$$(1-xu)^{-(s+1)}(1-yu)^{-(t+1)} = \alpha^{s+1}\sum_{j=0}^{t}\binom{j+s}{s}\beta^j(1-uy)^{-(t+1-j)} + \beta^{t+1}\sum_{k=0}^{s}\binom{k+t}{t}\alpha^k(1-ux)^{-(s+1-k)}. \tag{67}$$

Substituting (67) into (66) one get

$$\begin{aligned}
\mathbf{F}_1^{(\kappa_l)}(1, s+1, t+1; 2; x, y; b, d) &= \alpha^{s+1}\sum_{j=0}^{t}\binom{j+s}{s}\beta^j\int_0^1(1-uy)^{-(t+1-j)}\,\Theta\left(\kappa_l; -\frac{b}{u} - \frac{d}{1-u}\right) du \\
&\quad + \beta^{t+1}\sum_{k=0}^{s}\binom{k+t}{t}\alpha^k\int_0^1(1-ux)^{-(s+1-k)}\,\Theta\left(\kappa_l; -\frac{b}{u} - \frac{d}{1-u}\right) du \\
&= \alpha^{s+1}\sum_{j=0}^{t}\binom{j+s}{s}\beta^j\,{}_2F_1^{(\kappa_l)}\left[\begin{matrix}(t-j+1), 1\\ 2\end{matrix}; y; b, d\right] \\
&\quad + \beta^{t+1}\sum_{k=0}^{s}\binom{k+t}{t}\alpha^k\,{}_2F_1^{(\kappa_l)}\left[\begin{matrix}(s-k+1), 1\\ 2\end{matrix}; x; b, d\right] \\
&= y^{s+1}\sum_{j=0}^{t-1}\binom{j+s}{s}\frac{(-x)^j}{(y-x)^{j+s+1}}\,{}_2F_1^{(\kappa_l)}\left[\begin{matrix}(t-j+1), 1\\ 2\end{matrix}; y; b, d\right] \\
&\quad + x^{t+1}\sum_{k=0}^{s-1}\binom{k+t}{t}\frac{(-y)^k}{(y-x)^{k+t+1}}\,{}_2F_1^{(\kappa_l)}\left[\begin{matrix}(s-k+1), 1\\ 2\end{matrix}; x; b, d\right] \\
&\quad + \binom{t+s}{s}\frac{(-1)^t x^t y^s}{(y-x)^{t+s+1}}\left\{y\,{}_2F_1^{(\kappa_l)}\left[\begin{matrix}1, 1\\ 2\end{matrix}; y; b, d\right] + x\,{}_2F_1^{(\kappa_l)}\left[\begin{matrix}1, 1\\ 2\end{matrix}; x; b, d\right]\right\},
\end{aligned}$$

where we have applied the integral representation of extended Gauss hypergeometric function. □

Remark

If we set $b = d = 0$, then (65) is reduced to the following finite sum representation [8, p.215, Theorem 2.1a]

$$\begin{aligned}
\mathbf{F}_1(1, s+1, t+1; 2; x, y) &= \frac{(-x)^t y^s}{(y-x)^{s+t+1}}\binom{s+t}{s}[\ln(1-x) - \ln(1-y)] \\
&\quad - \sum_{j=0}^{t-1}\binom{j+s}{s}\frac{(-x)^j y^s\left[1-(1-y)^{j-t}\right]}{(y-x)^{s+j+1}(t-j)} \\
&\quad - \sum_{k=0}^{s-1}\binom{k+t}{t}\frac{x^t(-y)^k\left[1-(1-x)^{k-s}\right]}{(x-y)^{t+k+1}(s-k)}.
\end{aligned} \tag{68}$$

◊

Remark

This method can be used to find more complicated finite sum representations of extended Appell's hypergeometric series $\mathbf{F}_1^{(\kappa_l)}$. For instance, other finite sum representations given by [8, p.215, Theorem 2.1 b,c] can indeed be considered analogously in a simple and straightforward manner. ◊

It marks the end of the discussion of extended Appell's hypergeometric functions. Next, we will put focus on the Lauricell's hypergeometric functions. However, not all four Lauricella's functions can be generalized to their new forms.here, we will give a known generalized result.

Definition 3.2. *[18] Lauricella hypergeometric function* $\mathbf{F}_{D,(\kappa_l)}^{(r)}$ *is defined by*

$$\mathbf{F}_{D,(\kappa_l)}^{(r)}(\alpha,\beta_1,\cdots,\beta_r;\gamma;x_1,\cdots,x_r;b,d)$$
$$=\sum_{m_1,\cdots,m_r=0}^{\infty}(b_1)_{m_1}\cdots(b_r)_{m_r}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha+m_1+\cdots+m_r,\gamma-\alpha)}{B(\alpha,\gamma-\alpha)}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!}. \quad (69)$$

$$(\max\{|x_1|,\cdots,|x_r|\}<1;\min\{\Re(b),\Re(d)\}\geq 0)$$

Its Euler type integral representation of Lauricella hypergeometric function $\mathbf{F}_{D,(\kappa_l)}^{(r)}$ is stated in the following theorem.

Theorem 3.8. *For the extended Lauricella hypergeometric function* $\mathbf{F}_{D,(\kappa_l)}^{(r)}$ *defined by (69), the following integral integral representation hold true:*

$$\mathbf{F}_{D,(\kappa_l)}^{(r)}(\alpha,\beta_1,\cdots,\beta_r;\gamma;x_1,\cdots,x_r;b,d)$$
$$=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}\prod_{j=1}^{r}(1-x_jt)^{-\beta_j}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)dt. \quad (70)$$

$$(\Re(b),\Re(d)>0;b=d=0,\max\{|\arg(1-x_1)|,\cdots|\arg(1-x_r)|\}<\pi;\Re(\gamma)>\Re(\alpha)>0)$$

As a direct consequence of above theorem, we have the following summation formula

$$\mathbf{F}_{D,(\kappa_l)}^{(r)}(\alpha,\beta_1,\cdots,\beta_r;\gamma;1,\cdots,1;b,d)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha,\gamma-\alpha-\beta_1-\cdots-\beta_r).$$

If we put $x_1=x_2=\cdots=x_r=x$,

$$\mathbf{F}_{D,(\kappa_l)}^{(r)}(\alpha,\beta_1,\cdots,\beta_r;\gamma;x,\cdots,x;b,d)={}_2F_1^{(\kappa_l)}(\alpha;\beta_1+\cdots+\beta_r;\gamma;x;b,d).$$

Theorem 3.9. *One has*

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{r}(f_jt+g_j)^{\lambda_j}\Theta\left(\kappa_l;-\frac{p}{t-a}-\frac{q}{b-t}\right)dt=\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j}B(\alpha,\beta)$$
$$\times\mathbf{F}_{D,(\kappa_l)}^{(r)}\left(\alpha,-\lambda_1,\cdots,-\lambda_r;\alpha+\beta;-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_r}{af_r+g_r};\frac{p}{b-a},\frac{q}{b-a}\right) \quad (71)$$

where $a,b\in\mathbb{R}\ (a<b)$, $f_i,g_i,\lambda_i\in\mathbb{C},\ i=1,\cdots,k,\Re(b)>0,\Re(d)>0,\ \Re(\alpha)>0,\ \Re(\beta)>0$ *and*

$$\max\left\{\left|\frac{(b-a)f_1}{af_1+g_1}\right|,\cdots,\left|\frac{(b-a)f_r}{af_r+g_r}\right|\right\}<1.$$

Proof

Our proof mainly depends on the following expansion [17]

$$(f_jt+g_j)^{\lambda_j}=(af_j+g_j)^{\lambda_j}\sum_{m_j=0}^{\infty}\frac{(-\lambda_j)_{m_j}}{m_j!}\left(-\frac{(t-a)f_j}{af_j+g_j}\right)^{m_j}\quad(|(t-a)f_j|<|af_j+g_j|;\ t\in[a,\ b]). \quad (72)$$

Opening up $\prod_{j=1}^{r}(f_j t + g_j)^{\lambda_j}$ by using (72) and then integrating out t we have

$$\begin{aligned}
&\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{r}(f_j t+g_j)^{\lambda_j}\,\Theta\left(\kappa_l;-\frac{p}{t-a}-\frac{q}{b-t}\right)dt\\
&=\prod_{j=1}^{r}(af_j+g_j)^{\lambda_j}\sum_{m_1,\cdots m_r=0}^{\infty}\frac{(-\lambda_1)_{m_1}\cdots(-\lambda_r)_{m_r}}{m_1!\cdots m_r!}\left(-\frac{f_1}{af_1+g_1}\right)^{m_1}\cdots\left(-\frac{f_r}{af_r+g_r}\right)^{m_r}\\
&\times\int_a^b (t-a)^{\alpha+m_1+\cdots+m_r-1}(b-t)^{\beta-1}\,\Theta\left(\kappa_l;-\frac{p}{t-a}-\frac{q}{b-t}\right)dt. \qquad (73)
\end{aligned}$$

The last integral in (73) can be evaluate by applying substitution $t=(b-a)u+a$, namely,

$$\begin{aligned}
&\int_a^b (t-a)^{\alpha+m_1+\cdots+m_r-1}(b-t)^{\beta-1}\,\Theta\left(\kappa_l;-\frac{p}{t-a}-\frac{q}{b-t}\right)dt\\
&=(b-a)^{\alpha+\beta+m_1+\cdots+m_r-1}\int_0^1 u^{\alpha+m_1+\cdots+m_r-1}(1-u)^{\beta-1}\,\Theta\left(\kappa_l;-\frac{p}{(b-a)u}-\frac{q}{(b-a)(1-u)}\right)du\\
&=(b-a)^{\alpha+\beta+m_1+\cdots+m_r-1}\,\mathcal{B}^{(\kappa_l)}_{\frac{p}{b-a},\frac{q}{b-a}}(\alpha+m_1+\cdots+m_r,\beta). \qquad (74)
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{r}(f_j t+g_j)^{\lambda_j}\,\Theta\left(\kappa_l;-\frac{p}{t-a}-\frac{q}{b-t}\right)dt\\
&=(b-a)^{\alpha+\beta-1}\prod_{j=1}^{r}(af_j+g_j)^{\lambda_j}\sum_{m_1,\cdots m_r=0}^{\infty}(-\lambda_1)_{m_1}\cdots(-\lambda_r)_{m_r}\,\mathcal{B}^{(\kappa_l)}_{\frac{p}{b-a},\frac{q}{b-a}}(\alpha+m_1+\cdots+m_r,\beta)\\
&\times\frac{1}{m_1!}\left(-\frac{(b-a)f_1}{af_1+g_1}\right)^{m_1}\cdots\frac{1}{m_r!}\left(-\frac{(b-a)f_r}{af_r+g_r}\right)^{m_r}\\
&=(b-a)^{\alpha+\beta-1}\prod_{j=1}^{r}(af_j+g_j)^{\lambda_j}\,B(\alpha,\beta)\\
&\times\mathbf{F}^{(r)}_{D,(\kappa_l)}\left(\alpha,-\lambda_1,\cdots,-\lambda_r;\alpha+\beta;-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_r}{af_r+g_r};\frac{p}{b-a},\frac{q}{b-a}\right).
\end{aligned}$$

The second equality follows by the definition of extended Lauricella hypergeometric function $\mathbf{F}^{(r)}_{D,(\kappa_l)}$. □

Theorem 3.10.

$$\begin{aligned}
&\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^{r}t_j^{\beta_j-1}e^{-t_j}\,{}_1F_1^{(\kappa_l)}\left[\begin{matrix}\alpha\\ \gamma\end{matrix};x_1t_1+\cdots+x_rt_r;b,d\right]dt_1\cdots dt_r\\
&=\Gamma(\beta_1)\cdots\Gamma(\beta_r)\,\mathbf{F}^{(r)}_{D,(\kappa_l)}(\alpha,\beta_1,\cdots,\beta_r;\gamma;x_1,\cdots,x_r;b,d). \qquad (75)
\end{aligned}$$

$$(\Re(\beta_1),\cdots,\Re(\beta_r)>0)$$

Proof

The demonstration of this theorem depends on the application of the following series identity

$$\sum_{m_1,\cdots,m_r=0}^{\infty}f(m_1+\cdots+m_r)\frac{x_1^{m_1}\cdots x_r^{m_r}}{m_1!\cdots m_r!}=\sum_{m=0}^{\infty}f(m)\frac{(x_1+\cdots+x_r)^m}{m!}. \qquad (76)$$

Specifically, we have

$$\int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{r} t_j^{\beta_j-1} e^{-t_j} {}_1F_1^{(\kappa_l)}\left(\alpha;\gamma;x_1t_1++x_rt_r;b,d\right) dt_1\cdots dt_r$$

$$= \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{\mathcal{B}_{b,d}^{(\kappa_l)}\left(\alpha+m_1+\cdots+m_r,\gamma-\alpha\right)}{B\left(\alpha,\gamma-\alpha\right)} \frac{x_1^{m_1}\cdots x_r^{m_r}}{m_1!\cdots m_r!} \prod_{j=1}^{r}\int_0^\infty t_j^{\beta_j+m_j-1} e^{-t_j} dt_j$$

$$= \sum_{m_1,\cdots,m_r=0}^{\infty} \Gamma\left(\beta_1+m_1\right)\cdots\Gamma\left(\beta_r+m_r\right) \frac{\mathcal{B}_{b,d}^{(\kappa_l)}\left(\alpha+m_1+\cdots+m_r,\gamma-\alpha\right)}{B\left(\alpha,\gamma-\alpha\right)} \frac{x_1^{m_1}\cdots x_r^{m_r}}{m_1!\cdots m_r!}$$

$$= \Gamma\left(\beta_1\right)\cdots\Gamma\left(\beta_r\right) \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{\mathcal{B}_{b,d}^{(\kappa_l)}\left(\alpha+m_1+\cdots+m_r,\gamma-\alpha\right)}{B\left(\alpha,\gamma-\alpha\right)} \frac{x_1^{m_1}\cdots x_r^{m_r}}{m_1!\cdots m_r!}.$$

□

In accordance with the previous method, we can obtain the following generalization for classical Lauricella's hypergeometric function: $\mathbf{F}_A^{(r)}\left(\alpha,\beta_1,\cdots,\beta_r;\gamma_1,\cdots\gamma_r;x_1,\cdots,x_r\right)$.

Definition 3.3. *Lauricella hypergeometric function* $\mathbf{F}_A^{(r)}\left(\alpha,\beta_1,\cdots,\beta_r;\gamma_1,\cdots\gamma_r;x_1,\cdots,x_r\right)$ *is defined by*

$$\mathbf{F}_{A,(\kappa_l)}^{(r)}\left(\alpha,\beta_1,\cdots,\beta_r;\gamma_1,\cdots\gamma_r;x_1,\cdots,x_r;b,d\right) = \sum_{m_1,\cdots m_r=0}^{\infty} (\alpha)_{m_1+\cdots+m_r} \prod_{j=1}^{r} \frac{\mathcal{B}_{b,d}^{(\kappa_l)}\left(\beta_j+m_j,\gamma_j-\beta_j\right)}{B\left(\beta_j,\gamma_j-\beta_j\right)} \frac{x_1^{m_1}\cdots x_r^{m_r}}{m_1!\cdots m_r!}. \tag{77}$$

$$\left(|x_1|+\cdots+|x_r|<1;\ \min\{\Re(b),\Re(d)\}\geq 0\right)$$

When $b=d=0$, function (77) reduces to usual Lauricella hypergeometric function [16]:

$$\mathbf{F}_A^{(r)}\left(\alpha,\beta_1,\cdots,\beta_r;\gamma_1,\cdots\gamma_r;x_1,\cdots,x_r\right) = \sum_{m_1,\cdots m_r=0}^{\infty} (\alpha)_{m_1+\cdots+m_r} \frac{(b_1)_{m_1}\cdots(b_r)_{m_r}}{(c_1)_{m_1}\cdots(c_r)_{m_r}} \frac{x_1^{m_1}\cdots x_r^{m_r}}{m_1!\cdots m_r!}.$$

Comparing the coefficients of $\mathbf{F}_{A,(\kappa_l)}^{(r)}$ with those of $\mathbf{F}_A^{(r)}$, associated with **Definition 2.1**, it is clear that the construction of the coefficients of function (77) corresponds with what we generalize hypergeometric function.

For this new function, we have the following multidimensional integral representation.

Theorem 3.11. *For Lauricella hypergeometric function* $\mathbf{F}_A^{(r)}\left(\alpha,\beta_1,\cdots,\beta_r;\gamma_1,\cdots\gamma_r;x_1,\cdots,x_r\right)$*, we have*

$$\mathbf{F}_{A,(\kappa_l)}^{(r)}\left(\alpha,\beta_1,\cdots,\beta_r;\gamma_1,\cdots\gamma_r;x_1,\cdots,x_r;b,d\right) = \prod_{j=1}^{r} B\left(\beta_j,\gamma_j-\beta_j\right)$$
$$\times \int_0^1 \cdots \int_0^1 \prod_{j=1}^{r} u_j^{\beta_j-1}\left(1-u_j\right)^{\gamma_j-\beta_j-1} \Theta\left(\kappa_l;-\frac{b}{u_j}-\frac{d}{1-u_j}\right)\left(1-x_1u_1-\cdots-x_ru_r\right)^{-\alpha} du_1\cdots du_r. \tag{78}$$

$$\left(\Re(\beta_j)>0,\ \Re(\gamma_j-\beta_j)>0,\ j=1,\cdots,r;\ \min\{\Re(b),\Re(d)\}\geq 0\right)$$

Proof

This formula can be easily established by expanding the factor $\left(1-x_1u_1-\cdots-x_ru_r\right)^{-\alpha}$ as

$$\left(1-x_1u_1-\cdots-x_ru_r\right)^{-\alpha} = \sum_{m_1,\cdots,m_r=0}^{\infty} (\alpha)_{m_1+\cdots+m_r} \frac{(x_1u_1)^{m_1}\cdots(x_ru_r)^{m_r}}{m_1!\cdots m_r!}$$

and then integrating out $u_j,\ j=1,\cdots,r$ with the help of our extended beta integral (11). □

The following theorem show that extended Lauricella hypergeometric function $\mathbf{F}_{A,(\kappa_l)}^{(r)}$ can be expressed as a single integral whose integrand is a product of several extended Kummer hypergeometric functions ${}_1F_1^{(\kappa_l)}\left(\beta;\gamma;z;b,d\right)$.

Theorem 3.12.

$$\begin{aligned}
&\mathbf{F}^{(r)}_{A,(\kappa_l)}(\alpha,\beta_1,\cdots,\beta_r;\gamma_1,\cdots\gamma_r;x_1,\cdots,x_r;b,d)\\
&=\frac{1}{\Gamma(\alpha)}\int_0^1 e^{-t}t^{\alpha-1}{}_1F_1^{(\kappa_l)}\left[\begin{array}{c}\beta_1\\ \gamma_1\end{array};x_1t;b,d\right]\cdots{}_1F_1^{(\kappa_l)}\left[\begin{array}{c}\beta_r\\ \gamma_r\end{array};x_rt;b,d\right]dt.
\end{aligned}\tag{79}$$

Proof

Taking the series form of ${}_1F_1^{(\kappa_l)}(\beta_j;\gamma_j;x_jt;b,d)$, $j=1,\cdots,r$ and then integrating out t. □

By applying the relation $(\alpha)_{m_1+\cdots+m_{r-1}+m_r}=(\alpha)_{m_1+\cdots+m_{r-1}}(\alpha+m_1+\cdots+m_{r-1})_{m_r}$, we can write

$$\begin{aligned}
&\mathbf{F}^{(r)}_{A,(\kappa_l)}(\alpha,\beta_1,\cdots,\beta_r;\gamma_1,\cdots\gamma_r;x_1,\cdots,x_r;b,d)\\
&=\sum_{m_1,\cdots m_{r-1}=0}^{\infty}(\alpha)_{m_1+\cdots+m_{r-1}}\prod_{j=1}^{r-1}\frac{\mathcal{B}^{(\kappa_l)}_{b,d}(\beta_j+m_j,\gamma_j-\beta_j)}{B(\beta_j,\gamma_j-\beta_j)}\frac{x_1^{m_1}\cdots x_{r-1}^{m_{r-1}}}{m_1!\cdots m_{r-1}!}\\
&\quad\times{}_2F_1^{(\kappa_l)}\left(\begin{array}{c}\alpha+m_1+\cdots+m_{r-1},\beta_r\\ \gamma_r\end{array};x_r;b,d\right).
\end{aligned}\tag{80}$$

Theorem 3.13. *The Mellin-Barnes type integral representation of* $\mathbf{F}^{(r)}_{A,(\kappa_l)}$ *is*

$$\begin{aligned}
&\mathbf{F}^{(r)}_{A,(\kappa_l)}(\alpha,\beta_1,\cdots,\beta_r;\gamma_1,\cdots\gamma_r;x_1,\cdots,x_r;b,d)\\
&=\frac{1}{(2\pi i)^n}\int_{L_1}\cdots\int_{L_r}\frac{\Gamma(\alpha-s_1-\cdots-s_r)}{\Gamma(\alpha)}\prod_{j=1}^{r}\frac{\mathcal{B}^{(\kappa_l)}_{b,d}(\beta_j-s_j,\gamma_j-\beta_j)\Gamma(s_j)}{B(\beta_j,\gamma_j-\beta_j)}\\
&\times(-x_1)^{-s_1}\cdots(-x_r)^{-s_r}ds_1\cdots ds_r.
\end{aligned}\tag{81}$$

where $L_1,\cdots,L_r$ *are Barnes paths of integration.*

Proof

The basic ideas are the same when proving this theorem and **Theorem 3.4**. We take multidimensional Mellin transform of function $\mathbf{F}^{(r)}_{A,(\kappa_l)}(\alpha,\beta_1,\cdots,\beta_r;\gamma_1,\cdots\gamma_r;-x_1,\cdots,-x_r;b,d)$, get the result on the basis of applying the *method of Brackets* and take the inverse Mellin transform of the function we get.

To obtain the multidimensional Mellin transform of $\mathbf{F}^{(r)}_{A,(\kappa_l)}(\alpha,\beta_1,\cdots,\beta_r;\gamma_1,\cdots\gamma_r;-x_1,\cdots,-x_r;b,d)$, we multiply both sides of (77) by $x_1^{s_1-1}\cdots x_r^{s_r-1}$ and integrate with respect to $x_1,\cdots,x_r$ over $[0,\infty)\times\cdots\times[0,\infty)$. One has

$$\begin{aligned}
&\int_0^\infty\cdots\int_0^\infty x_1^{s_1-1}\cdots x_r^{s_r-1}\mathbf{F}^{(r)}_{A,(\kappa_l)}(\alpha,\beta_1,\cdots,\beta_r;\gamma_1,\cdots\gamma_r;-x_1,\cdots,-x_r;b,d)\,dx_1\cdots dx_r\\
&=\sum_{m_1,\cdots m_r=0}^{\infty}\frac{\Gamma(\alpha+m_1+\cdots+m_r)}{\Gamma(\alpha)}\prod_{j=1}^{r}\frac{\mathcal{B}^{(\kappa_l)}_{b,d}(\beta_j+m_j,\gamma_j-\beta_j)}{B(\beta_j,\gamma_j-\beta_j)}\frac{(-1)^{m_1}\cdots(-1)^{m_r}}{m_1!\cdots m_r!}\\
&\times\int_0^\infty\cdots\int_0^\infty x_1^{m_1+s_1-1}\cdots x_r^{m_r+s_1-1}dx_1\cdots dx_r\\
&=\sum_{m_1,\cdots m_r=0}^{\infty}f(m_1,\cdots,m_r)\,\phi_{m_1}\cdots\phi_{m_r}\langle s_1+m_1\rangle\cdots\langle s_r+m_r\rangle.
\end{aligned}$$

The result here is just more complicated in form than that of equation (56). And with the basic ideas when we prove **Theorem 3.4** and with the applying of **Rule III**[1], we can get

$$\begin{aligned}
&\int_0^\infty\cdots\int_0^\infty x_1^{s_1-1}\cdots x_r^{s_r-1}\mathbf{F}^{(r)}_{A,(\kappa_l)}(\alpha,\beta_1,\cdots,\beta_r;\gamma_1,\cdots\gamma_r;-x_1,\cdots,-x_r;b,d)\,dx_1\cdots dx_r\\
&\qquad=\frac{\Gamma(\alpha-s_1-\cdots-s_r)}{\Gamma(\alpha)}\prod_{j=1}^{r}\frac{\mathcal{B}^{(\kappa_l)}_{b,d}(\beta_j-s_j,\gamma_j-\beta_j)\Gamma(s_j)}{B(\beta_j,\gamma_j-\beta_j)}.
\end{aligned}$$

Taking multidimensional inverse Mellin transform we have

$$
\begin{aligned}
&\mathbf{F}^{(r)}_{A,(\kappa_l)}(\alpha,\beta_1,\cdots,\beta_r;\gamma_1,\cdots\gamma_r;-x_1,\cdots,-x_r;b,d)\\
&=\frac{1}{(2\pi i)^n}\int_{L_1}\cdots\int_{L_r}\frac{\Gamma(\alpha-s_1-\cdots-s_r)}{\Gamma(\alpha)}\prod_{j=1}^{r}\frac{\mathcal{B}^{(\kappa_l)}_{b,d}(\beta_j-s_j,\gamma_j-\beta_j)\Gamma(s_j)}{B(\beta_j,\gamma_j-\beta_j)}\\
&x_1^{-s_1}\cdots x_r^{-s_r}ds_1\cdots ds_r.
\end{aligned}
$$

Theorem 3.13 follows from replacing $-x_i, i=1,\cdots,r$ with x_i. □

The way of proving the following two theorems are exactly the same.

Theorem 3.14. *The Mellin-Barnes type integral representation of* $\mathbf{F}^{(r)}_{D,(\kappa_l)}$ *is*

$$
\begin{aligned}
&\mathbf{F}^{(r)}_{D,(\kappa_l)}(\alpha,\beta_1,\cdots,\beta_r;\gamma;x_1,\cdots,x_r;b,d)=\frac{1}{(2\pi i)^n}\\
&\times\int_{L_1}\cdots\int_{L_r}\prod_{j=1}^{r}\frac{\Gamma(\beta_j-s_j)\Gamma(s_j)}{\Gamma(\beta_j)}\frac{\mathcal{B}^{(\kappa_l)}_{b,d}(\alpha-s_1-\cdots-s_r,\gamma-\alpha)}{B(\alpha,\gamma-\alpha)}(-x_1)^{-s_1}\cdots(-x_r)^{-s_2}ds_1\cdots ds_r
\end{aligned}
\tag{82}
$$

where $L_1,\cdots,L_r$ *are Barnes paths of integration.*

Theorem 3.15. *For* $p=q+1$*, we have*

$$
\begin{aligned}
&\Gamma(\alpha_1)\prod_{j=1}^{q}\frac{\Gamma(\alpha_{j+1})\Gamma(\beta_j-\alpha_{j+1})}{\Gamma(\beta_j)}{}_{q+1}F_q^{(\kappa_l)}\left(\begin{matrix}\alpha_1&\cdots&\alpha_{q+1}\\\beta_1&\cdots&\beta_q\end{matrix};-(x_1+\cdots+x_r);b,d\right)\\
&=\frac{1}{(2\pi i)^n}\int_{L_1}\cdots\int_{L_r}\Gamma(\alpha_1-s_1-\cdots-s_r)\prod_{j=1}^{q}\mathcal{B}^{(\kappa_l)}_{b,d}(\alpha_{j+1}-s_1-\cdots-s_r,\beta_j-\alpha_{j+1})\\
&\times\Gamma(s_1)\cdots\Gamma(s_r)x_1^{-s_1}\cdots x_r^{-s_r}ds_1\cdots ds_r
\end{aligned}
\tag{83}
$$

For $p=q$

$$
\begin{aligned}
&\prod_{j=1}^{q}\frac{\Gamma(\alpha_j)\Gamma(\beta_j-\alpha_j)}{\Gamma(\beta_j)}{}_qF_q^{(\kappa_l)}\left(\begin{matrix}\alpha_1&\cdots&\alpha_q\\\beta_1&\cdots&\beta_q\end{matrix};-(x_1+\cdots+x_r);b,d\right)\\
&=\frac{1}{(2\pi i)^n}\int_{L_r}\cdots\int_{L_r}\prod_{j=1}^{q}\mathcal{B}^{(\kappa_l)}_{b,d}(\alpha_j-s_1-\cdots-s_r,\beta_j-\alpha_j)\\
&\times\Gamma(s_1)\cdots\Gamma(s_r)x_1^{-s_1}\cdots x_r^{-s_r}ds_1\cdots ds_r
\end{aligned}
\tag{84}
$$

For $p<q$

$$
\begin{aligned}
&\frac{1}{\Gamma(\beta_1)\cdots\Gamma(\beta_r)}\prod_{j=1}^{p}\frac{\Gamma(\alpha_j)\Gamma(\beta_{r+j}-\alpha_j)}{\Gamma(\beta_{r+j})}{}_pF_q^{(\kappa_l)}\left(\begin{matrix}\alpha_1&\cdots&\alpha_p\\\beta_1&\cdots&\beta_q\end{matrix};-(x_1+\cdots+x_r);b,d\right)\\
&=\frac{1}{(2\pi i)^n}\int_{L_1}\cdots\int_{L_r}\prod_{j=1}^{r}\frac{1}{\Gamma(\beta_j-s_1-\cdots-s_r)}\prod_{j=1}^{p}\mathcal{B}^{(\kappa_l)}_{b,d}(\alpha_j-s_1-\cdots-s_r,\beta_{r+j}-\alpha_j)\\
&\times\Gamma(s_1)\cdots\Gamma(s_r)x_1^{-s_1}\cdots x_r^{-s_r}ds_1\cdots ds_r
\end{aligned}
\tag{85}
$$

where $L_1,\cdots,L_r$ *are Barnes paths of integration.*

Remark

Note that when parameters $b=d=0$, integral (83), (84) and (85) reduce to

$$
\begin{aligned}
&\frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}{}_pF_q\left[\begin{matrix}\alpha_1&\cdots&\alpha_p\\\beta_1&\cdots&\beta_q\end{matrix};-(x_1+\cdots+x_r)\right]\\
&=\frac{1}{(2\pi i)^n}\int_{L_1}\cdots\int_{L_r}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j+s_1+\cdots+s_r)}{\prod_{j=1}^{q}\Gamma(\beta_j+s_1+\cdots+s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}ds_1\cdots ds_r
\end{aligned}
\tag{86}
$$

where the contours are Barnes type with indentations, if necessary, such that the poles of $\Gamma(\alpha_j + s_1 + \cdots + s_r)$, $j = 1, \cdots, p$ are separated from those of $\Gamma(-s_j)$, $j = 1, \cdots, r$. And this multiple integral representation for the generalized hypergeometric series is given by Saigo and Saxena [16, p. 212]. ♢

4. Applications in Hilbert-Hardy type Inequalities

We have introduced many different generalizations of usual hypergeometric functions, such as, extended generalized hypergeometric function, Appell's and Lauricella's hypergeometric functions. A lot of important results are established in the previous sections. Now we hope to find some applications in other branches of mathematics. In this section, we will use extended Gauss hypergeometric function to establish several important inequalities.

Let $f(x), g(y) \geq 0$, $f(x) \in L^p(0, +\infty)$, $g(x) \in L^q(0, +\infty)$, $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. Then we have the following two equivalent inequalities as

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y} dx dy \leq \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{\int_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(y)\,dy\right\}^{\frac{1}{q}} \tag{87}$$

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx\right)^p dy \leq \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p \int_0^\infty f^p(x)\,dx \tag{88}$$

where the constant factor $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ and $\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p$ are the best possible in (87) and (88) respectively. And the equality in (87) and (88) holds iff $f(x) = 0$ or $g(x) = 0$. Inequality (87) is called Hardy-Hilbert integral inequality (see [4], [14]). During the last decade inequality (87) and (88) were generalized in many different ways, some of the generalizations rely heavily on the integral identities involving hypergeometric functions and beta functions (see [4], [14]). In [14], the author obtain some new Hardy-Hilbert type inequalities with fractional kernels, which involve the constants expressed in terms of Gauss hypergeometric functions. So it is natural to consider whether these new inequalities can be extended to the more general case by using our extended Gauss hypergeometric functions. The answer is positive. In the whole process of proving, we can get some previously unimagined properties. Note that in what follows, we chose function: $\Theta(\kappa_l; z) = \exp(z)$. This expression are easy to handle and the final form of our result will thus be more simple.

We need the following two integral identities.

Lemma 2. *Suppose* $a, b, c, \alpha, \gamma, \tilde{p}, \tilde{q} \in \mathbb{R}$ *are s.t.* $\tilde{p} \geq 0, \tilde{q} \geq 0$, $a + c > b > 0$ *and* $0 < \alpha < 2\gamma$. *Then*

$$\int_0^\infty \frac{x^{b-1}}{(1+\gamma x)^c (1+\alpha x)^a} \exp\left(-\gamma \tilde{q} x - \frac{\tilde{p}\gamma^{-1}}{x}\right) dx = e^{\tilde{p}+\tilde{q}} \gamma^{-b} B(b, c+a-b)\, {}_2F_1\left[\begin{array}{c} a, b \\ c+a \end{array}; \frac{\gamma-\alpha}{\gamma}; \tilde{p}, \tilde{q}\right] \tag{89}$$

$$\int_0^\infty \frac{x^{b-1}}{(1+\alpha x)^a (1+\gamma x)^c} \exp\left(-\alpha \tilde{q} x - \frac{\tilde{p}\alpha^{-1}}{x}\right) dx = e^{\tilde{p}+\tilde{q}} \alpha^{-b} B(b, c+a-b)\, {}_2F_1\left[\begin{array}{c} c, b \\ c+a \end{array}; \frac{\alpha-\gamma}{\alpha}; \tilde{p}, \tilde{q}\right] \tag{90}$$

Proof

We start with the integral representation (13) with $\Theta(\kappa_l; z) = \exp(z)$, i.e.,

$$\int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} \exp\left(-\frac{\tilde{p}}{t} - \frac{\tilde{q}}{1-t}\right) dt = B(b, c-b)\, {}_2F_1\left[\begin{array}{c} a, b \\ c \end{array}; z; \tilde{p}, \tilde{q}.\right] \tag{91}$$

By using the substitutions $1 - t = \frac{1}{1+u}$ in (91), we have

$$\int_0^\infty u^{b-1} (1+u)^{a-c} (1+u-zu)^{-a} \exp\left(-u\tilde{q} - \frac{\tilde{p}}{u}\right) du = e^{\tilde{p}+\tilde{q}} B(b, c-b)\, {}_2F_1\left[\begin{array}{c} a, b \\ c \end{array}; z; \tilde{p}, \tilde{q}\right] \tag{92}$$

Then we substitute γx for u in Eq (92)

$$\int_0^\infty \frac{x^{b-1}}{(1+\gamma x)^{c-a} (1+(\gamma - z\gamma) x)^a} \exp\left(-\gamma \tilde{q} x - \frac{\tilde{p}\gamma^{-1}}{x}\right) du = e^{\tilde{p}+\tilde{q}} \gamma^{-b} B(b, c-b)\, {}_2F_1\left[\begin{array}{c} a, b \\ c \end{array}; z; \tilde{p}, \tilde{q}\right] \tag{93}$$

The first assertion of **lemma 2** follows from setting $\gamma(1-z)=\alpha$ and replace $c-a$ with c in above equation. To prove the second assertion we can make transformation $u=\alpha x$ in Eq. (92) and then set $\alpha(1-z)=\gamma$. Replacing $c-a$ with c in the result and then interchange a and c we obtain the relation (90). $\square$

Remark

If we let $\tilde{q}=\tilde{p}=0$, **lemma 2** reduce to [14, LEMMA 1.]

$$\int_0^\infty \frac{x^{b-1}}{(1+\alpha x)^a(1+\gamma x)^c}dx=\gamma^{-b}B(b,a+c-b)F\left[\begin{array}{c}a,b\\a+c\end{array};\frac{\gamma-\alpha}{\gamma}\right].$$

In fact, introducing the factor $\exp\left(-\frac{\tilde{p}}{t}-\frac{\tilde{q}}{1-t}\right)$ weakens the symmetry of integrand. So we need two integral identities to establish our results. $\diamondsuit$

It is time to finish the proof of **Theorem 2.5**.

The Proof of Theorem 2.5: With the help of the formulas (89) and (90) of **Lemma 2** we are able to finish the proof of the Euler transformation mentioned in Theorem 2.5. Let us set $\tilde{q}=\frac{\gamma\tilde{q}}{\alpha}$ and $\tilde{p}=\frac{\alpha\tilde{p}}{\gamma}$ in the second assertion (90). Then (90) becomes

$$\int_0^\infty \frac{x^{b-1}}{(1+\alpha x)^a(1+\gamma x)^c}\exp\left(-\gamma\tilde{q}x-\frac{\tilde{p}\gamma^{-1}}{x}\right)dx=e^{\frac{\alpha\tilde{p}}{\gamma}+\frac{\gamma\tilde{q}}{\alpha}}\alpha^{-b}B(b,c+a-b)\,{}_2F_1\left[\begin{array}{c}c,b\\c+a\end{array};\frac{\alpha-\gamma}{\alpha};\frac{\alpha\tilde{p}}{\gamma},\frac{\gamma\tilde{q}}{\alpha}\right]. \tag{94}$$

By equalizing the identities (94) and (89) we obtain the relation:

$$e^{-\left(\frac{\frac{\gamma}{\alpha}-1}{\frac{\gamma}{\alpha}}\right)\tilde{p}-\left(1-\frac{\gamma}{\alpha}\right)\tilde{q}}\left(\frac{\gamma}{\alpha}\right)^b{}_2F_1\left[\begin{array}{c}c,b\\c+a\end{array};1-\frac{\gamma}{\alpha};\frac{\alpha\tilde{p}}{\gamma},\frac{\gamma\tilde{q}}{\alpha}\right]={}_2F_1\left[\begin{array}{c}a,b\\c+a\end{array};\frac{\frac{\gamma}{\alpha}-1}{\frac{\gamma}{\alpha}};\tilde{p},\tilde{q}\right]. \tag{95}$$

If we write $\frac{\gamma}{\alpha}=1-z$, then (93) becomes

$$e^{-(1-z)\tilde{p}-z\tilde{q}}(1-z)^b{}_2F_1\left[\begin{array}{c}c-a,b\\c\end{array};z;\frac{\tilde{p}}{1-z},(1-z)\tilde{q}\right]={}_2F_1\left[\begin{array}{c}a,b\\c\end{array};\frac{-z}{1-z};\tilde{p},\tilde{q}\right]. \tag{96}$$

By using the notation occured in **section 2**. we have:

$$e^{-(1-z)b-zd}(1-z)^{\alpha_2}{}_2F_1\left[\begin{array}{c}\beta_1-\alpha_1,\alpha_2\\\beta_1\end{array};z;\frac{b}{1-z},(1-z)d\right]={}_2F_1\left[\begin{array}{c}\alpha_1,\alpha_2\\\beta_1\end{array};\frac{-z}{1-z};b,d\right] \tag{97}$$

Substituing equation (95) into (24) we get:

$$\begin{aligned}{}_2F_1\left[\begin{array}{c}\alpha_1,\alpha_2\\\beta_1\end{array};z;b,d\right]&=(1-z)^{-\alpha_1}{}_2F_1\left[\begin{array}{c}\alpha_1,\beta_1-\alpha_2\\\alpha_2\end{array};\frac{-z}{1-z};d,b\right]\\&=e^{-(1-z)b-zd}(1-z)^{\beta_1-\alpha_2-\alpha_1}{}_2F_1\left[\begin{array}{c}\beta_1-\alpha_1,\beta_1-\alpha_2\\\beta_1\end{array};z;\frac{b}{1-z},(1-z)d\right]\end{aligned} \tag{98}$$

This is in fact a generalized Euler transformation.

Let us come back to the proof of inequalities. Before we state and prove our new inequality, we need to define some weighted functions. We define $F:(0,\infty)\mapsto\mathbb{R}$ by:

$$F(x;\tilde{p},\tilde{q})=\left[\int_0^\infty \frac{y^{-q'A_2}}{(x+\alpha_1 y)^{s_1}(x+\alpha_2 y)^{s_2}}\exp\left(-\alpha_1\tilde{q}\frac{y}{x}-\frac{\tilde{p}}{\alpha_1}\frac{x}{y}\right)dy\right]^{\frac{1}{q'}}, \tag{99}$$

where $\tilde{p}\geq 0$, $\tilde{q}\geq 0$, $\mu>0$, $\alpha_1,\alpha_2>0$, $\frac{1}{2}<\frac{\alpha_1}{\alpha_2}<2$, $s_1+s_2>0$ and $A_2\in\left(\frac{1-s_1-s_2}{q'},\frac{1}{q'}\right)$. Define $G:(0,\infty)\mapsto\mathbb{R}$ by:

$$G(y;\tilde{p},\tilde{q})=\left[\int_0^\infty \frac{x^{-p'A_1}}{(x+\alpha_1 y)^{s_1}(x+\alpha_2 y)^{s_2}}\exp\left(-\frac{\tilde{q}}{\alpha_2}\frac{x}{y}-\frac{\tilde{p}}{\alpha_2}\frac{y}{x}\right)dx\right]^{\frac{1}{p'}}, \tag{100}$$

where $\tilde{p}\geq 0$, $\tilde{q}\geq 0$, $\mu>0,\alpha_1,\alpha_2>0$, $\frac{1}{2}<\frac{\alpha_1}{\alpha_2}<2$, $s_1+s_2>0$ and $A_1\in\left(\frac{1-s_1-s_2}{p'},\frac{1}{p'}\right)$. By virtue of **lemma 1** we can compute $F(x)$ and $G(y)$ and express them as extended Gauss hypergeometric functions.

Lemma 3. *Suppose $\mu>0,\alpha_1,\alpha_2>0$, $\frac{1}{2}<\frac{\alpha_1}{\alpha_2}<2$, $s_1+s_2>0$, Further, let A_1 and A_2 be real parameters*

such that $A_1 \in \left(\frac{1-s_1-s_2}{p'}, \frac{1}{p'}\right)$ *and* $A_2 \in \left(\frac{1-s_1-s_2}{q'}, \frac{1}{q'}\right)$. *If the functions* F *and* G *are defined by (99) and (100) respectively, then*

$$F(x;\tilde{p},\tilde{q}) = e^{\frac{\tilde{p}+\tilde{q}}{q'}} k(F;\ \tilde{p},\tilde{q})\, x^{\frac{1-s_1-s_2}{q'}-A_2}, \tag{101}$$

$$G(y;\tilde{p},\tilde{q}) = e^{\frac{\tilde{p}+\tilde{q}}{p'}} k(G;\ \tilde{p},\tilde{q})\, y^{\frac{1-s_1-s_2}{p'}-A_1}, \tag{102}$$

where

$$k(F;\ \tilde{p},\tilde{q}) = \alpha_1^{A_2-\frac{1}{q'}} B^{\frac{1}{q'}}(1-q'A_2, s_1+s_2+q'A_2-1)\left\{{}_2F_1\left[\begin{array}{c} s_2, 1-q'A_2 \\ s_1+s_2 \end{array}; \frac{\alpha_1-\alpha_2}{\alpha_1}; \tilde{p},\tilde{q}\right]\right\}^{\frac{1}{q'}}, \tag{103}$$

$$k(G;\ \tilde{p},\tilde{q}) = \alpha_1^{-\frac{s_1}{p'}} \alpha_2^{\frac{1-s_2}{p'}-A_1} B^{\frac{1}{p'}}(1-p'A_1, s_1+s_2+p'A_1-1)\left\{{}_2F_1\left[\begin{array}{c} s_1, 1-p'A_1 \\ s_1+s_2 \end{array}; \frac{\alpha_1-\alpha_2}{\alpha_1}; \tilde{p},\tilde{q}\right]\right\}^{\frac{1}{p'}}. \tag{104}$$

Proof

This lemma is a direct consequence of **lemma 2**. □

Now we are prepared to derive our main result.

Theorem 4.1. *Let* p *and* q *be the real parameters such that* $p > 1,\ q > 1,\ \frac{1}{p}+\frac{1}{q} \geq 1$,*and let* p' *and* q' *respectively be their conjugate exponents, that is,* $\frac{1}{p}+\frac{1}{p'} = 1$ *and* $\frac{1}{q}+\frac{1}{q'} = 1$. *Define* $\lambda = \frac{1}{p'}+\frac{1}{q'}$ *and note that* $0 < \lambda \leq 1$, *for all* p *and* q. *Further, suppose* $\mu > 0, \alpha_1, \alpha_2 > 0, \frac{1}{2} < \frac{\alpha_1}{\alpha_2} < 2, s_1+s_2 > 0$. *If* f *and* g *are non-negative measurable functions on* $(0,\infty)$, *then the following inequalities:*

$$\int_0^\infty\int_0^\infty \frac{\exp\left[-(\alpha_1\tilde{q}+\alpha_2\tilde{p})\left(\frac{y}{x}+\frac{1}{\alpha_1\alpha_2}\frac{x}{y}\right)\right]}{(x+\alpha_1 y)^{\lambda s_1}(x+\alpha_2 y)^{\lambda s_2}} f(x)\, g(y)\, dxdy$$
$$\leq e^{2(\tilde{p}+\tilde{q})} K_{\tilde{p}.\tilde{q}} \left(\int_0^\infty x^{\frac{p}{q'}(1-s_1-s_2)+p(A_1-A_2)} f^p(x)\, dx\right)^{\frac{1}{p}} \left(\int_0^\infty y^{\frac{q}{p'}(1-s_1-s_2)+q(A_2-A_1)} g^q(y)\, dy\right)^{\frac{1}{q}} \tag{105}$$

and

$$\left[\int_0^\infty y^{\frac{q'}{p'}(s_1+s_2-1)+q'(A_1-A_2)}\left(\int_0^\infty \frac{\exp\left[-(\alpha_1\tilde{q}+\alpha_2\tilde{p})\left(\frac{y}{x}+\frac{1}{\alpha_1\alpha_2}\frac{x}{y}\right)\right]}{(x+\alpha_1 y)^{\lambda s_1}(x+\alpha_2 y)^{\lambda s_2}} f(x)\, dx\right)^{q'} dy\right]^{\frac{1}{q'}}$$
$$\leq e^{2(\tilde{p}+\tilde{q})} K_{\tilde{p}.\tilde{q}} \left(\int_0^\infty x^{\frac{p}{q'}(1-s_1-s_2)+p(A_1-A_2)} f^p(x)\, dx\right)^{\frac{1}{p}} \tag{106}$$

hold for any $A_1 \in \left(\frac{1-s_1-s_2}{p'}, \frac{1}{p'}\right)$ *and* $A_2 \in \left(\frac{1-s_1-s_2}{q'}, \frac{1}{q'}\right)$. *The constant*

$$K_{\tilde{p}.\tilde{q}} = k(F;\ q'\tilde{p}, q'\tilde{q})\, k(G;\ p'\tilde{p}, p'\tilde{q}).$$

Inequalities (105) and (106) are equivalent. In addition, the equality in (105) and (106) holds iff at least one of the functions f *or* g *is equal to zero.*

Proof

The left-hand side of (105) can be written as

$$\int_0^\infty\int_0^\infty \frac{\exp\left[-(\alpha_1\tilde{q}+\alpha_2\tilde{p})\left(\frac{y}{x}+\frac{1}{\alpha_1\alpha_2}\frac{x}{y}\right)\right]}{(x+\alpha_1 y)^{\lambda s_1}(x+\alpha_2 y)^{\lambda s_2}} f(x)\, g(y)\, dxdy = \int_0^\infty\int_0^\infty K_1^{\frac{1}{q'}}(x,y)\, K_2^{\frac{1}{p'}}(x,y)\, K_3^{1-\lambda}(x,y)\, dxdy \tag{107}$$

where

$$K_1(x,y) = \underbrace{\frac{F^{p-q'}(x; q'\tilde{p}, q'\tilde{q})\, f^p(x)}{(x+\alpha_1 y)^{s_1}(x+\alpha_2 y)^{s_2}} \frac{x^{pA_1}}{y^{q'A_2}}}_{I_1(x,y)} \exp\left(-\alpha_1 q'\tilde{q}\frac{y}{x} - q'\frac{\tilde{p}\alpha_1^{-1}}{yx^{-1}}\right)$$

$$K_2(x,y)=\underbrace{\frac{G^{q-p'}(y;p'\tilde{p},p'\tilde{q})\,g^q(y)}{(x+\alpha_1 y)^{s_1}(x+\alpha_2 y)^{s_2}}\frac{y^{qA_2}}{x^{p'A_1}}}_{I_2(x,y)}\exp\left(-\alpha_2^{-1}p'\tilde{q}\frac{x}{y}-p'\frac{\tilde{p}\alpha_2}{xy^{-1}}\right)$$

$$K_3(x,y)=x^{pA_1}F^p(x;q'\tilde{p},q'\tilde{q})\,y^{qA_2}G^q(y;p'\tilde{p},p'\tilde{q})\,f^p(x)\,g^q(y)\,.$$

(It is easy but time-consuming to verify Eq (107).) and the functions F and G are defined by (99), (100). Now, since the exponents satisfy identity: $\frac{1}{p'}+\frac{1}{q'}+(1-\lambda)=1$. We can use Holder's inequality, namely,

$$\begin{aligned}
&\int_0^\infty\int_0^\infty K_1^{\frac{1}{q'}}(x,y)\,K_2^{\frac{1}{p'}}(x,y)\,K_3^{1-\lambda}(x,y)\,dxdy\\
&\le\left(\int_0^\infty\int_0^\infty K_1(x,y)\,dxdy\right)^{\frac{1}{q'}}\left(\int_0^\infty\int_0^\infty K_2(x,y)\,dxdy\right)^{\frac{1}{p'}}\left(\int_0^\infty\int_0^\infty K_3(x,y)\,dxdy\right)^{1-\lambda}\\
&\le\left(\int_0^\infty x^{pA_1}F^p(x;q'\tilde{p},q'\tilde{q})\,f^p(x)\,dx\right)^{\frac{1}{q'}}\left(\int_0^\infty y^{qA_2}G^q(y;p'\tilde{p},p'\tilde{q})\,g^q(y)\,dy\right)^{\frac{1}{p'}}\\
&\quad\times\left[\left(\int_0^\infty y^{qA_2}G^q(y;p'\tilde{p},p'\tilde{q})\,g^q(y)\,dy\right)\left(\int_0^\infty x^{pA_1}F^p(x;q'\tilde{p},q'\tilde{q})\,f^p(x)\,dx\right)\right]^{1-\lambda}\\
&\le\left\{e^{\frac{p}{q'}(q'\tilde{p}+q'\tilde{q})}k^p(F;q'\tilde{p},q'\tilde{q})\right\}^{\frac{1}{p}}\left\{e^{\frac{q}{p'}(p'\tilde{p}+p'\tilde{q})}k^q(G;p'\tilde{p},p'\tilde{q})\right\}^{\frac{1}{q}}\\
&\quad\times\left(\int_0^\infty x^{\frac{p}{q'}(1-s_1-s_2)+p(A_1-A_2)}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{\frac{q}{p'}(1-s_1-s_2)+q(A_2-A_1)}g^q(y)\,dy\right)^{\frac{1}{q}}.
\end{aligned}$$

where

$$\left\{e^{\frac{p}{q'}(q'\tilde{p}+q'\tilde{q})}k^p(F;q'\tilde{p},q'\tilde{q})\right\}^{\frac{1}{p}}\left\{e^{\frac{q}{p'}(p'\tilde{p}+p'\tilde{q})}k^q(G;p'\tilde{p},p'\tilde{q})\right\}^{\frac{1}{q}}=e^{2(\tilde{p}+\tilde{q})}k(F;q'\tilde{p},q'\tilde{q})\,k(G;p'\tilde{p},p'\tilde{q})\,.$$

It marks the end of the proving of the first part of the theorem. The equivalence of (105) and (106) and the conditions for equality to hold can be proved in the way which explained in [14, p. 649, Theorem 1]. □

Remark

In fact, the way we prove the inequalities (105) and (106) provides a common method to generalize inequalities related to gauss hypergeometric functions. Once some integral identities are established, we can prove it according to the primary way of proving. For instance, the inequalities in the reference [4] can be generalize in the same way. (we should notice that we use **definition 2.2** instead of **2.1**).

On another aspect,we can directly use the definition of extended hypergeometric functions to construct inequalities.because the new definition is strong and flexible enough to get the desired results. ♢

5. Conclusion

By using extended Beta functions we have defined some extensions of generalized hypergeometric function and Lauricella's hypergeometric function. In some special cases, those extensions have been studied in [7], [21] and [22]. With flexible manipulating of extended definitions, we have obtained some generalization of properties of hypergeometric functions in classical theories. We have also explained an example of Hardy-Hilbert type inequality involving extended Gauss hypergeometric functions in detail, which may be a precedent for applying these extension in other branches of mathematics. It is worthwhile to point out that some fundamental relations between fractional calculus and extended generalized hypergeometric functions have been proposed in this paper. If we want to study it in the aspect of fractional calculus, we need to get more properties of the fractional operator defined by (43). As far as this is concerned, we have gotten some ideas, which will be presented in next publications.

Acknowledgement

The author is grateful to the anonymous referee for his/her valuable comments and suggestions on the improvement of this paper.

References

[1] T. Amdeberhan, O. Espinosa, I. Gonzalez, M. Harrison, V. H. Moll and A. Straub, Ramanujan's Master Theorem, Ramanujan J (2012) 29:103-120.

[2] L.C. Andrews, Special Functions for Engineers and Applied Mathematicians, New York: Mc Millan, 1985.

[3] G. E. Andrews, R. Askey, and R. Roy, Special Functions, Cambridge Univ. Press, (1999).

[4] L. E. Azar, Some Extensions of Hilbert's Integral Inequality, Journal of Math. Ineq., Vol 5, Number 1(2011), 131-140.

[5] M.A.Chaudhry, S.M.Zubair, Generalized incomplete gamma functions with applications, J. Comput. Appl. Math. 55 (1994) 99-124.

[6] M.A.Chaudhry, A.Qadir, M.Rafique, S.M.Zubair, Extension of Euler's beta function, J. Comput. Appl. Math. 78 (1997) 19-32.

[7] M.A.Chaudhry, Asghar Qadir, H.M.Srivastava, R.B.Paris, Extended hypergeometric and confluent hypergeometric functions, Appl. Math.Comput. 159 (2) (2004) 589–602.

[8] A. Cuyt, K. Driver, J. Tan, B. Vertonk, A finite sum representation of the Appell series $F_1(a, b, b; c; x, y)$, J. Comput. Appl. Math. 5 (1999) 213–219.

[9] K. A. Driver and S. J. Johnston, An Integral Representation of Some Hypergeometric Functions, ETNA, Vol 25, pp. 115-120, 2006.

[10] A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Higher Transcendental Functions, 1, McGraw-Hill, New York, (1953).

[11] I. Gonzalez, V. Moll, Definite integrals by the method of brackets. Part 1. Adv. Appl. Math. (2009).

[12] C. M. Joshi, V. Vyas, Extensions of certain classical integrals of Erdélyi for Gauss hypergeometric functions, J. Comput. Appl. Math. 160 (2003), 125-138.

[13] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V, Amsterdam, 2006

[14] Mario Krnic, Hilbert Inequality and Gaussian Hypergeometric Functions, Journal of Math. Ineq., 3, 4 (2009), 645-656.

[15] A. M. Mathai and Hans J. Haubold, Special Functions for Applied Scientists, Springer, New York, (2008).

[16] A. M. Mathai, R. K. Saxena and H. J. Haubold, The H-Function: Theory and Application, Springer, New York, (2009).

[17] H. M. Srivastava and M. A. Husssain, Fractional Integration of the H-Function of Several Variables, Computers Math. Applic. Vol. 30, No. 9, pp. 73-85, 1995.

[18] H. M. Srivastava, R. K. Parmar and P. Chopra, A Class of Extended Fractional Derivative Operators and Associated Generating Relations Involving Hypergeometric Functions, Axioms 2012, I, 238-258.

[19] S.B. Opps, N. Saad, H.M. Srivastava, Some reduction and transformation formulas for the Appell hypergeometric function F_2 , J. Math. Anal. Appl. 302 (2005) 180-195.

[20] S.B. Opps, N. Saad, H.M. Srivastava, Recursion formulas for Appell's hypergeometric function F_2 with some applications to radiation field problems, Appl. Math. Comput. 207 (2009) 545-558

[21] Ozarslan, M.A.; Ozergin, E. Some generating relations for extended hypergeometric function via generalized fractional derivative operator. Math. Comput. Model. 2010, 52, 1825-1833.

[22] E. Ozergin, M.A. Ozarslan, A. Altin, Extension of gamma, beta and hypergeometric functions, J. Comput. Appl. Math. 2011, 235, 4601šC4610.

[23] E. D. Rainville, Special Functions, Macmillan, New York, 1960.

[24] Z. X. Wang and D.R. Guo, Special functions, World Scientific.1989.